Homework - II (Probability tutorial)

Algorithmical and Statistical Modeling, Fall 2012.
Due: Oct. 25, Thursday

1. Let $f : [0,2] \to [0,\infty]$ be a function defined on a measure space $([0,2], \mathcal{B}([0,2])$ such that $f[0,1] = \infty$ and $f(1,2] = 2$. Exhibit nonnegative measurable simple functions

$$s_1 < s_2 < \cdots < f$$

such that $s_n(x) \to f(x)$ for every $x \in [0,2]$. (Note that a simple function is real-valued hence cannot attain infinite values) **(20 points)**

2. Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$, where m is the Lebesgue measure. Give examples of measurable functions f defined on $\mathbb{R}$ satisfying :

   (a) $f(x) = 0$ if x is a rational number and $\int f dm = 1/2$.

   (b) $-\infty < f(x) < \infty$, but $\int f dm$ is not defined . **(30 points)**

3. Find the cdf of the random variable X defined on $([0,2], \mathcal{B}([0,2]), P)$, where $P(A) = \frac{m(A)}{2}$ for all $A \in \mathcal{B}([0,2])$ (m is the Lebesgue measure) and

$$X(\omega) := \begin{cases} 1 & : \text{ if } 0 \leq \omega < 1, \\ 2 & : \text{ if } 1 \leq \omega \leq 2. \end{cases}$$ **(25 points)**

4. Let X and Y be two random variables defined on $(\Omega, \mathcal{F}, P)$ such that $X = Y$ almost everywhere. Suppose the probability density function (pdf) of X exists, shows that the pdf of Y also exists. **(25 points)**