Final Exam (Tutorial)

Algorithmical and Statistical Modeling, Fall 2012.

State whether the **assertion** or **conclusion** in the each set of statements itself is either "true" or "false" by giving justification **as briefly as possible** — in the justification you can derive elementary facts from the statements or give examples to show that they can justify/falsify the statement. Points are awarded to a correct certification only depending on the justification that follows it. For some statements you "may" consider the hints provided.

Warning : Pay careful attention to *each word* in the given statement before you write out your justifications. Also, the justifications could be very simple and even one-liners. If the justification is longer than four or five sentences, then you are not concise.

(Each problem carries 6 points).

1. On a measurable space $(\Omega, \mathcal{F})$, subsets of Ω, A_1 and A_2 are such that $A_1 \cup A_2 \in \mathcal{F}$. Then $A_1 \in \mathcal{F}$.

2. On some measure space $(\Omega, \mathcal{F}, \mu)$ the characteristic function χ_A for any set $A \in \mathcal{F}$ satisfies the equality
$$\int_\Omega \chi_A d\mu \ + \ \int_\Omega \chi_{A^c} d\mu = 1,$$
where A^c denotes the complement of A relative to Ω. Then μ is a probability measure.

3. Two real valued random variables X and Y defined on $(\Omega, \mathcal{F}, P)$ are such that that $X(\omega) < Y(\omega)$ for all $\omega \in \Omega$. Then their joint cdf satisfies the relation
$$F_{XY}(a, a) = F_X(a) \text{ for every } a \in \mathbb{R}.$$

4. Two random variables X and Y defined on a probability space $(\Omega, \mathcal{F}, P)$ are such that $X(\omega) < Y(\omega) < \infty$ for all $\omega \in \Omega$. Assume the variances of X and Y are well defined. Then
$$\text{Var}(X) < \text{Var}(Y).$$

5. Consider a probability space $(\Omega, \mathcal{F}, P)$ and some $A, B \in \mathcal{F}$. Let χ_* denote the characteristic function of the set $*$. If χ_A and χ_B are given to be independent random variables then χ_{A^c} and χ_B are also independent random variables – here as usual A^c denotes the complement of A relative to Ω. (You may use the **Hint:** Compute $\sigma(\chi_A)$, $\sigma(\chi_B)$ and $\sigma(\chi_{A^c})$ and infer).

6. Let $X_1, X_2, X_3, \ldots$ be a sequence of zero mean random variables be such that $\text{Var}(\frac{1}{n}\sum_{i=1}^n X_i)$ tends to 0 in the limit as $n \to \infty$. Then the random variable $\frac{1}{n}\sum_{i=1}^n X_i$ converges in probability to a constant 0 as $n \to \infty$.

Solutions.

1. **False.** Consider a measurable space $(\Omega, \mathcal{F})$ such that Ω has only two elements a_1 and a_2 with $\mathcal{F}$ being the trivial sigma-algebra $\{\emptyset, \Omega\}$. Clearly $\{a_1\} \cup \{a_2\} = \Omega$, but $\{a_1\} \notin \mathcal{F}$.

2. **True.** We need to verify only whether $\mu(\Omega) = 1$. Let $A = \Omega$. Then

$$\begin{aligned} \int_\Omega \chi_\Omega d\mu + \int_\Omega \chi_\emptyset d\mu &= 1, & (1)\\ \Rightarrow \quad 1 \cdot \mu(\Omega) + 0 &= 1. & (2) \end{aligned}$$

3. **Question had a typo: It should have been $F_{XY}(a,a) = F_Y(a)$.** By definition, $F_{XY}(a,a) = P(\{\omega : X(\omega) \leq a, Y(\omega) \leq a\})$. In other words, $F_{XY}(a,a) = P\Big(\{\omega : X(\omega) \leq a\} \cap \{\omega : Y(\omega) \leq a\}\Big)$. Since $Y(\omega) > X(\omega)$, $\{\omega : X(\omega) \leq a\} \supseteq \{\omega : Y(\omega) \leq a\}$. Hence $F_{XY}(a,a) = P(\{\omega : Y(\omega) \leq a\}) = F_Y(a)$. (**Note: If you have made attempt with the right initial step, full marks will be awarded**).

4. **False.** Take $Y = X + 1$. Both X and Y have the same variance.

5. **True.** $\sigma(\chi_A) = \{\emptyset, A, A^c, \Omega\}$ and $\sigma(\chi_B) = \{\emptyset, B, B^c, \Omega\}$. Since X and Y are independent, $\sigma(\chi_A)$ and $\sigma(\chi_B)$ are independent. But $\sigma(\chi_{A^c}) = \sigma(\chi_A)$. Hence $\sigma(\chi_{A^c})$ and $\sigma(\chi_B)$ are independent which means χ_{A^c} and χ_B are independent.

6. **True.** Let $\epsilon > 0$. Since mean of $\frac{1}{n}\sum_{i=1}^n X_i$ is 0, by Chebyschev inequality:

$$P\left(\left|\frac{1}{n}\sum_{i=1}^n X_i\right| \geq \epsilon\right) \leq \frac{\text{Var}(\frac{1}{n}\sum_{i=1}^n X_i)}{\epsilon^2}.$$

Since $\text{Var}(\frac{1}{n}\sum_{i=1}^n X_i) \to 0$ as $n \to \infty$, $|\frac{1}{n}\sum_{i=1}^n X_i|$ converges in probability to 0 by definition.