Homework - I (Probability tutorial)

Algorithmical and Statistical Modeling, Fall 2012.
Due: Oct. 2, Tuesday (to be submitted in the Lecture)

1. Exhibit two real valued sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n \;>\; \limsup_{n\to\infty}(x_n + y_n). \qquad \textbf{(5 points)}$$

2. If $A_1, A_2, \ldots$ are elements belonging to a sigma-algebra $\mathcal{F}$ then show that the set

$$\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$$

belongs to $\mathcal{F}$. **(10 points)**

3. Let $\Omega = \{a, b, c\}$. Exhibit two sigma algebras on Ω such that their union is not a sigma-algebra. **(15 points)**

4. Let μ be a measure on a sigma-algebra $\mathcal{F}$.

   (a) Let $A_1 \subset A_2 \subset A_3 \subset \cdots$. If $A = \bigcup_{i=1}^{\infty} A_i$ show that $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.

   (b) Let $A_1 \supset A_2 \supset A_3 \supset \cdots$. If $A = \bigcap_{i=1}^{\infty} A_i$ show that $\lim_{n\to\infty} \mu(A_n) = \mu(A)$. **(30 points)**

5. Let $\Omega = \{a, b, c, d\}$. Let $S = \{1, 2, 3\}$. Construct (nontrivial) sigma algebras $\mathcal{F}$ and $\mathcal{B}$ on Ω and S respectively such that $f : \Omega \to S$ is measurable. Also construct a $g : \Omega \to S$ which is not measurable (here, a sigma-algebra is trivial if it has just the null set and the whole space). **(15 points)**

6. Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ and $(\Omega_3, \mathcal{F}_3)$ be measurable spaces. If $f : \Omega_1 \to \Omega_2$ and $f : \Omega_2 \to \Omega_3$ are respectively $(\mathcal{F}_1, \mathcal{F}_2)$ and $(\mathcal{F}_2, \mathcal{F}_3)$ measurable functions, prove that $f_2 \circ f_1 : \Omega_1 \to \Omega_3$, where $f_2 \circ f_1(x) := f_2(f_1(x))$ is $(\mathcal{F}_1, \mathcal{F}_3)$ measurable. **(10 points)**

7. Let $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$ be measurable spaces and let $f : \Omega_1 \to \Omega_2$ be a measurable function. Then for any measure μ on $(\Omega_i, \mathcal{F}_i)$, verify the function on Ω_2 given by

$$\lambda(A) := \mu(f^{-1}(A)), \qquad A \in \mathcal{F}_2$$

is a measure on $\mathcal{F}_2$. The measure λ is called the induced measure (or the measure induced by f). Show that if μ is a probability measure then so is λ. **(15 points)**