Biomedical Signal Processing

Data Engineering Program, spring 2018

Exercise Sheet 1: Solutions

March 10, 2018

Exercise 1) Simple filters

1-1. A discrete- time signal $x[n]$ is shown in Figure P1-1. Sketch and label the following signals.

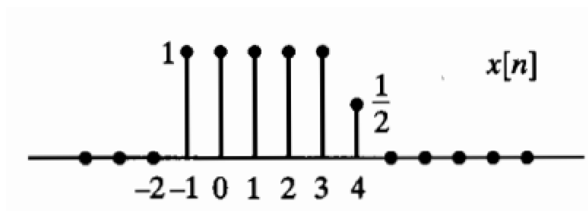
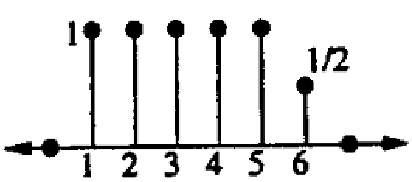
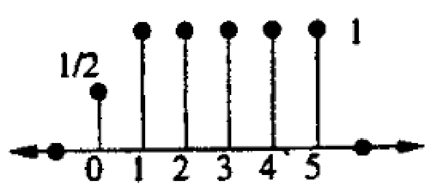


Figure P1-1.

1-1.a $x[n-2]$

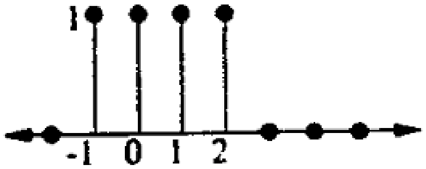


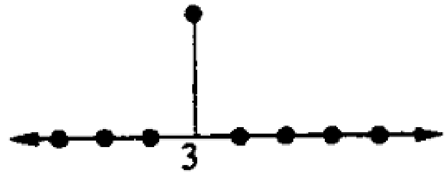
1-1.b $x[4-n]$



1-1.c $x[2n]$

1-1.d $x[n-1]\delta[n-3]$

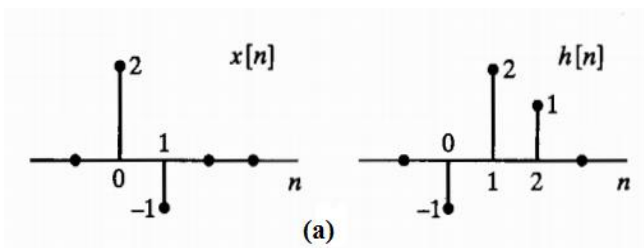




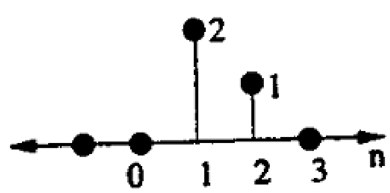
Exercise 2) Convolution sum

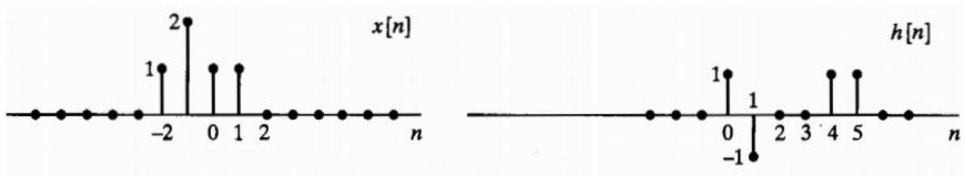
2-1. For each of the pairs of sequences in Figure P2-1, use discrete convolution to find the response to the input $x[n]$ of the linear time-invariant system with impulse response $h[n]$.

Recall that for the LTI systems, $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$. So, applying the graphical approach, the convolution sums will be as follows:



$y[n] = \delta[n-1] * h[n] = h[n-1]$

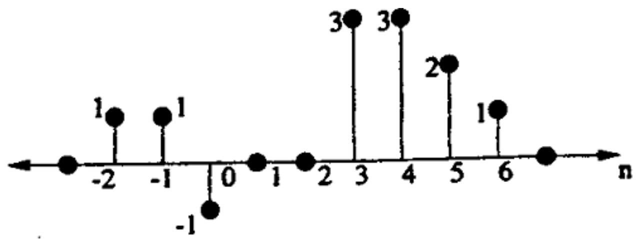




(b)

Figure P2-1.

$$y[n] = (\delta[n+2] + 2\delta[n+1] + \delta[n] + \delta[n-1]) * h[n]$$
$$= h[n+2] + 2h[n+1] + h[n] + h[n-1]$$



Exercise 3) Frequency Response of LTI systems

3-1. Indicate which of the following discrete-time signals are eigenfunctions of stable, linear time-invariant discrete-time systems:

3-1-a. $e^{2\pi nj/3}$ ✓	3-1-b. 3^n ✓	3-1-c. $2^n u[-n-1]$
3-1-d. $\cos(\omega_0 n)$	3-1-e. $(1/4)^n$ ✓	3-1-f. $(1/4)^n u[n] + 4^n u[-n-1]$

3-2-a. Find the frequency response of the LTI system whose input and output satisfy the difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

If we take the Fourier Transform of two sides, we get:

$$Y(e^{j\omega})\left(1 - \frac{1}{2}e^{-j\omega}\right) = X(e^{j\omega})\left(1 + 2e^{-j\omega} + e^{-2j\omega}\right).$$

Then:

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{(1 + 2e^{-j\omega} + e^{-2j\omega})}{\left(1 - \frac{1}{2}e^{-j\omega}\right)}$$

3-2-b. write a difference equation that characterizes a system whose frequency response is:

$$H(e^{j\omega}) = \frac{1-\frac{1}{2}e^{-j\omega}+e^{-3j\omega}}{1+\frac{1}{2}e^{-j\omega}+\frac{3}{4}e^{-2j\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

After cross multiplying:

$$\left(1+\frac{1}{2}e^{-j\omega}+\frac{3}{4}e^{-2j\omega}\right)Y(e^{j\omega}) = (1-\frac{1}{2}e^{-j\omega}+e^{-3j\omega})X(e^{j\omega}).$$

Then applying the inverse transform we get:

$$y[n]+\frac{1}{2}y[n-1]+\frac{3}{4}y[n-2] = x[n]-\frac{1}{2}x[n-1]+x[n-3]$$

3-3. a. Determine the Fourier transform of the sequence

$$r[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & otherwise \end{cases}.$$

Looking at Figure 3-10 in the lecture notes, we get:

$$R(e^{j\omega}) = e^{-jM\omega/2}\frac{\sin[\omega(M+1)/2]}{\sin\omega/2}$$

Or we can write:

$$R(e^{j\omega}) = \sum_{n=0}^{M} e^{-j\omega n} = \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} = e^{-jM\omega/2}\left(\frac{e^{\frac{j\omega(M+2)}{2}}-e^{\frac{-j\omega(M+2)}{2}}}{e^{j\omega}-e^{-j\omega}}\right) = e^{-jM\omega/2}\frac{\sin[\omega(M+1)/2]}{\sin\omega/2}$$

3-3. b. Consider the sequence

$$w[n] = \begin{cases} \frac{1}{2}\left[1-\cos(\frac{2\pi n}{M})\right] & 0 \le n \le M \\ 0 & otherwise \end{cases},$$

express the $W(e^{j\omega})$, the Fourier transform of $w[n]$, in terms of $R(e^{j\omega})$, the Fourier transform of $r[n]$.

We can write $w[n] = r[n].\frac{1}{2}\left[1+\cos(\frac{2\pi n}{M})\right]$, therefore, $W(e^{j\omega}) = R(e^{j\omega}) * F\{\frac{1}{2}\left[1+\cos(\frac{2\pi n}{M})\right]\}$ where $F\{\frac{1}{2}\left[1+\cos(\frac{2\pi n}{M})\right]\}$ is Fourier transform of $\frac{1}{2}\left[1+\cos(\frac{2\pi n}{M})\right]$.

$$F\left\{\frac{1}{2}\left[1+\cos\left(\frac{2\pi n}{M}\right)\right]\right\} = \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[1+\cos\left(\frac{2\pi n}{M}\right)\right] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[1+\frac{1}{2}e^{\frac{2\pi n}{M}j}+\frac{1}{2}e^{-\frac{2\pi n}{M}j}\right] e^{-j\omega}$$

$$= \frac{1}{2}\left[\delta(\omega)+\frac{1}{2}\,\delta\left(\omega+\frac{2\pi}{M}\right)-\frac{1}{2}\delta\left(\omega-\frac{2\pi}{M}\right)\right].$$

Thus,

$$W(e^{j\omega}) = R(e^{j\omega}) * \frac{1}{2}\left[\delta(\omega)+\frac{1}{2}\,\delta\left(\omega+\frac{2\pi}{M}\right)-\frac{1}{2}\delta\left(\omega-\frac{2\pi}{M}\right)\right]$$

Exercise 4) The output of the LTI system

4-1. Determine the output of a linear time-invariant system if the impulse response $h[n]$ and the input $x[n]$ are as follows:

a) $x[n] = u[n]$ and $h[n] = a^n u[-n-1]$, with $a > 1$

$$y[n] = a^n u[-n-1] * u[n] = \sum_{k=-\infty}^{\infty} a^k u[-k-1]u[n-k] = \begin{cases} \sum_{k=-\infty}^{n} a^k & n \leq -1 \\ \sum_{k=-\infty}^{-1} a^k & n > 1 \end{cases}$$

$$= \begin{cases} \frac{a^n}{1-a} & n \leq -1 \\ \frac{1/a}{1-1/a} & n > 1 \end{cases}$$

b) $x[n] = u[n-4]$ and $h[n] = 2^n u[-n-1]$

From part (a) we know $2^n u[-n-1] * u[n] = \begin{cases} 2^{n+1} & n \leq -1 \\ 1 & n > 1 \end{cases}$, then as the system is time invariant, $2^n u[-n-1] * u[n-4] = \begin{cases} 2^{n-3} & n \leq 3 \\ 1 & n > 3 \end{cases}$.

4-2. Consider an LTI system with frequency response

$$H(e^{j\omega}) = e^{-j(\omega-\frac{\pi}{4})} \frac{1+e^{-j2\omega}+4e^{-j4\omega}}{1+\frac{1}{2}e^{-j2\omega}}, \qquad -\pi < \omega < \pi.$$

Determine the output $y[n]$ for all n if the input for all n is $x[n] = \cos(\frac{\pi n}{2})$.

$x[n] = \cos\left(\frac{\pi n}{2}\right) = \frac{e^{j\frac{\pi n}{2}} + e^{-j\frac{\pi n}{2}}}{2}$ and $X(e^{j\omega}) = \pi(\delta\left(\omega + \frac{\pi}{2}\right) + \delta\left(\omega - \frac{\pi}{2}\right))$

then,

$$Y(e^{j\omega}) = H(e^{j\omega}).\pi\left(\delta\left(\omega + \frac{\pi}{2}\right) + \delta\left(\omega - \frac{\pi}{2}\right)\right) = \pi H\left(e^{-j\frac{\pi}{2}}\right) + \pi H\left(e^{j\frac{\pi}{2}}\right) =$$

$$\pi(e^{-j\left(-\frac{\pi}{2}-\frac{\pi}{4}\right)} \frac{1+e^{j2\frac{\pi}{2}}+4e^{j4\frac{\pi}{2}}}{1+\frac{1}{2}e^{j2\frac{\pi}{2}}} + e^{-j\left(\frac{\pi}{2}-\frac{\pi}{4}\right)} \frac{1+e^{-j2\frac{\pi}{2}}+4e^{-j4\frac{\pi}{2}}}{1+\frac{1}{2}e^{-j2\frac{\pi}{2}}})$$

After applying some mathematical simplifications and inverse Fourier transform, we get:

$$y[n] = 8e^{j\frac{\pi}{4}} \cos\left(\frac{\pi}{2}n - \frac{\pi}{2}\right)$$

4-3. Consider the system in Figure P4-2.

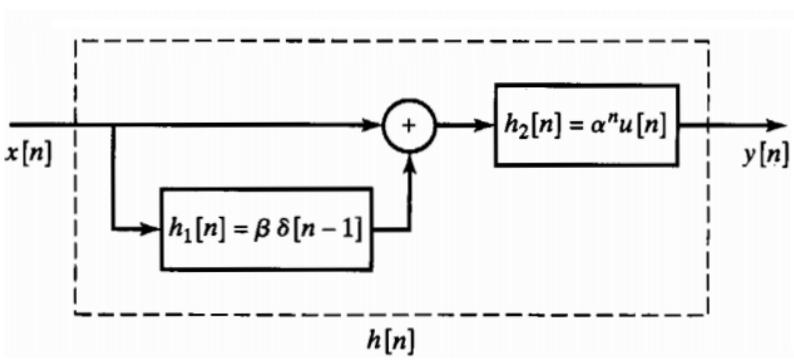


Figure P4-2.

(a) Find the impulse response $h[n]$ of the overall system.

As stated in the class, $y[n] = (x[n] + x[n] * h_1[n]) * h_2[n] = (x[n] * (\delta[n] + h_1[n])) * h_2[n]$. If $h[n]$ is the impulse response of the overall system such that $y[n] = x[n] * h[n]$, then $h[n] = (\delta[n] + h_1[n]) * h_2[n] = h_2[n] + h_1[n] * h_2[n] = \alpha^n u[n] + \beta\delta[n-1] * \alpha^n u[n] = \alpha^n u[n] + \beta * \alpha^{n-1} u[n-1]$

(b) Find the frequency response of the overall system.

Regarding the Fourier transform pairs listed in Figure 3-10 and property 2 in Table 3-3:

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} + \frac{\beta e^{-j\omega}}{1-\alpha e^{-j\omega}} = \frac{1-\beta e^{-j\omega}}{1-\alpha e^{-j\omega}} \quad for\ |\alpha| < 1$$

(c) Specify a difference equation that relates the output $y[n]$ to the input $x[n]$.

From part (b):

$$H(e^{j\omega}) = \frac{1-\beta e^{-j\omega}}{1-\alpha e^{-j\omega}} = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \quad for\ |\alpha| < 1,$$

by a cross multiplication, we get

$(1 - \alpha e^{-j\omega})\, Y(e^{j\omega}) = (1 - \beta e^{-j\omega})\, X(e^{j\omega})$.

Now, taking the inverse Fourier transform, the system equation would be:

$$y[n] - \alpha y[n-1] = x[n] - \beta x[n-1]$$

(d) Is this system causal? Yes, form part (a), $h[n] = 0$ for $n < 0$

Under what condition would the system be stable? The system is stable if it has a Fourier transform for which $|\alpha| < 1$ needs to be satisfied