3.10 z- Transform

The z- transform of a sequence $x[n]$ is define as:

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (3\text{-}60)$$

where z is a continuous complex variable and the equation is, in general, an infinite sum or an infinite power series. The transformation, $\mathcal{Z}\{.\}$ which maps the discrete sequence, $x[n]$, into a continuous function $X(z)$ is called *z-transform*. The correspondence between the sequence and its z-transform is shown by the notation:

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \quad (3\text{-}61)$$

In contrast to *two-sided* or *bilateral* z-transform described by Eq. (3-60), *one-sided* or *unilateral* z-transform is defined as:

$$X(z) = Z\{x[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n} \quad (3\text{-}62)$$

If $x[n] = 0$ for all $n < 0$, it is obvious that two-sided and one-sided z-transforms are equivalent. In this course, we only take the two-sided transformation into account.

In general, z is a complex variable such as $z = re^{j\omega}$; hence, $X(z)$ is also a complex variable and it is more convenient to represent and interpret it in complex polar plane (see Figure 3-11-a).

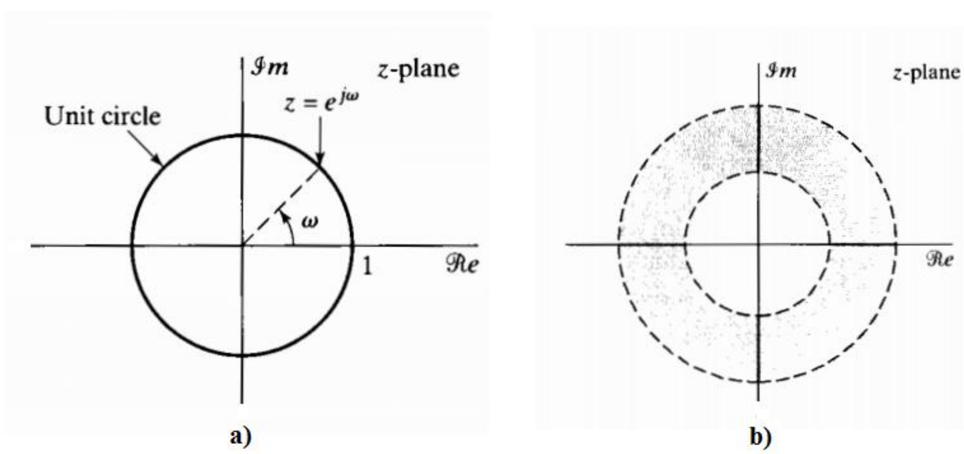

Figure 3-11: a) Complex z-plane, b) Region of convergence as a ring in complex z-plane [1].

If we compare the Eq. (3-60) and Eq. (3-47), it is evident that the z-transform reduces to Fourier transform if $z = e^{j\omega}$. The contour corresponding to $|z| = 1$ in z-plane is a circle of unit radius which is known as unit circle. Therefore, the z-transform evaluated on this circle corresponds to the Fourier transform. On the complex z-plane, ω is the angle between the vector point to z on the unit circle and the real axis. Starting from $\omega = 0$ $(z = 1)$ and moving through $\omega = \frac{\pi}{2}$ $(z = j)$ and then to $\omega = \pi$ $(z = -1)$, we obtain the Fourier transform for $0 \leq \omega \leq \pi$. If we continue moving around the unit circle, it would correspond to examining the Fourier transform from $\omega = \pi$ to $\omega = 2\pi$ or equivalently, from $\omega = 0$ to $\omega = -\pi$. This interpretation reveals why the frequency of the Fourier transform is inherently periodic. Indeed, a change of angle of 2π radians in the z-plane means traversing the unit circle once and returning to exactly the same point.

3.10.1 Region of Convergence (ROC)

As it was mentioned in section 3.7, the infinite sum representing the Fourier transform may not be finite for some sequences. It means, the Fourier transform does not converge for those sequences. Similarly, the z-transform may not converge for some sequences or for some values of z. For any arbitrary sequence, the set of values of z for which the z-transform converges is called the Region of Convergence (ROC). To ensure the convergence of the z-transform, the following condition must be satisfied:

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty; \qquad (3\text{-}63)$$

therefore, the ROC is defined as the set of values of z such that the Eq. (3-63) holds. With this definition, if some value of z, say $z = z_0$, is in the ROC, it means all values of z on the circle of radius $|z| = |z_0|$ will be in the ROC. Consequently, the region of convergence consists of a ring in the complex z-plane centered about the origin (see Figure 3-11-b). The outer boundary of ROC will be a circle or it may extend outward to infinity. The inner boundary of ROC, similarly, will be a circle or it may extend inward to include the origin. It is worth mentioning that it is possible for the z-transform to converge even if the Fourier transform does not. However, if the ROC of the z-transform for a sequence includes the unit circle, the Fourier transform of the sequence converges.

The z-transform is practically useful when the infinite sum can be expressed in closed form as a simple mathematical formula. Among the most important and useful z-transforms are those for which $X(z)$ is a rational function inside the region of convergence, i.e.,

$$X(z) = \frac{P(z)}{Q(z)} \tag{3-64}$$

where $P(z)$ and $Q(z)$ are polynomials in z. The values of z for which $X(z) = 0$ are called the zeros of $X(z)$, and the values of z for which $X(z)$ is infinite are referred to as the poles of $X(z)$. The poles of $X(z)$ for finite values of z are the roots of the denominator polynomial. In addition, poles may occur at $z = 0$ or $z = \infty$. For rational z-transforms, a number of important relationships exist between the locations of poles of $X(z)$ and the region of convergence of the z-transform.

Example 3-11: Right side and left-side exponential sequences

a) The signal $x[n] = a^n u[n]$ is non-zero for $n \geq 0$, hence, this is a right side sequence. From the Eq. (3-60), its z-transform is:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n . \tag{3-65}$$

In order to make sure that $X(z)$ converges, we require that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty. \tag{3-66}$$

Therefore, the ROC consists of the ranges of values of z for which $|az^{-1}| < 1$, or equivalently, $|z| > |a|$. Inside the ROC, the infinite series converge to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad for\ |z| > |a|. \tag{3-67}$$

In this example, if $|a| < 1$, the ROC includes the unit circle and the Fourier transform exists for $x[n] = a^n u[n]$.

Figure 3-12-a illustrates the pole-zero plot and the ROC for $x[n] = a^n u[n]$.

b) The signal $x[n] = -a^n u[-n-1]$ is non-zero only for $n \leq -1$, hence, this is a left side sequence. From the Eq. (3-60), its z-transform is:

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{n=1}^{\infty} (az^{-1})^{-n} = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n . \tag{3-68}$$

Now, if the $|(a^{-1}z)| < 1$ or, equivalently, $|z| < |a|$, the sum in Eq. (3-68) converges

$$X(z) = 1 - \sum_{n=0}^{\infty}(a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad for\ |z| < |a|. \qquad (3\text{-}69)$$

The pole-zero plot and the ROC for $x[n] = -a^n u[-n-1]$ are depicted in Figure 3-12-b.

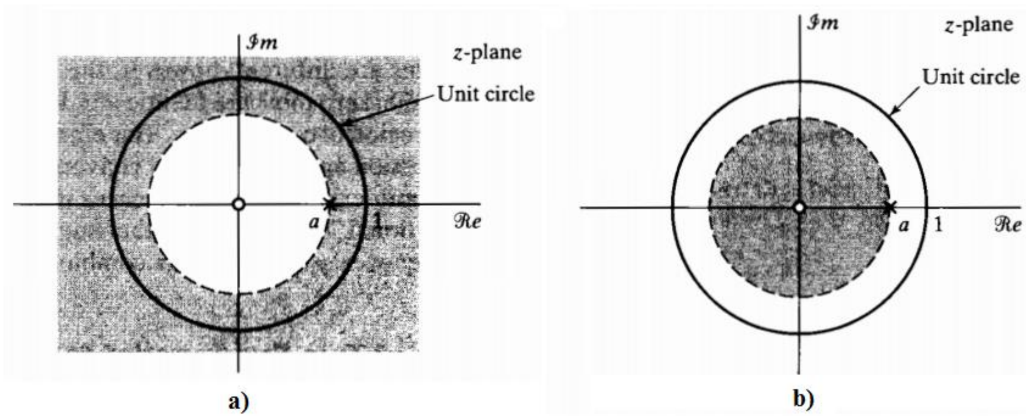

Figure 3-12: Pole-zero plot and ROC for sequences in example 3-11 a) and b) [1].

Example 3-12: Sum of two exponential sequences, $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$

Using the general result of Example 3-11-a with $a = \left(\frac{1}{2}\right)$, $a = \left(-\frac{1}{3}\right)$, the z-transform for two individual terms are:

$$\left(\frac{1}{2}\right)^n u[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{1}{1-\frac{1}{2}z^{-1}} \qquad for\ |z| > \frac{1}{2}. \qquad (3\text{-}70)$$

$$\left(-\frac{1}{3}\right)^n u[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{1}{1+\frac{1}{3}z^{-1}} \qquad for\ |z| > \frac{1}{3}. \qquad (3\text{-}71)$$

Therefore,

$$\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \qquad for\ |z| > \frac{1}{2}. \qquad (3\text{-}72)$$

In Figure 3-13: zero-pole plots and the ROC for two individual terms and for the sum of two exponential sequences are shown.

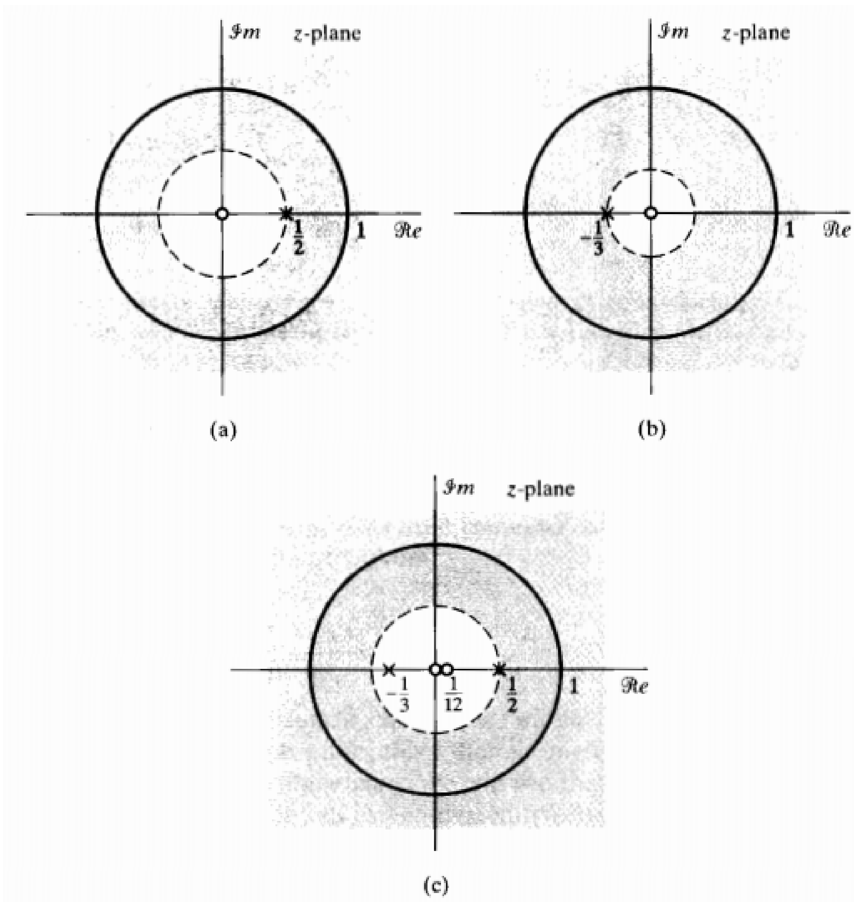

Figure 3-13: zero-pole plots and the ROC for two individual terms and for the sum of two exponential sequences in example 3-12 [1].

Figure 3-14 provides a list of commonly used z-transform pairs.

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$

7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $[\cos \omega_0 n]u[n]$	$\frac{1-[\cos \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1-[r\cos \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2 z^{-2}}$	$\|z\| > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r\sin \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2 z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Figure 3-14: a list of commonly used z-transform pairs [1].

ROC for the z-transform have some properties which mainly depend on the nature of the signal. If $x[n]$ has finite amplitude (except possibly in $n = \mp\infty$) and $X(z)$ can be written as a rational function, properties listed in Table 3-4 hold for the ROC.

Table 3-4 Properties of ROC for z-transform.

Property 1	The ROC is a ring or disk in the z-plane centered at the origin
Property 2	The Fourier transform of $x[n]$ converges if and only if the ROC of the z-transform of $x[n]$ includes the unit circle.
Property 3	The ROC doesn't contain any poles.
Property 4	If x[n] is a finite-duration sequence, i.e., a sequence that is zero except in a finite interval (i.e., $-\infty < N_1 \le n \le N_2 < \infty$), then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.
Property 5	If $x[n]$ is a right-sided sequence, the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.
Property 6	If $x[n]$ is a left-sided sequence, the ROC extends inward from the innermost (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.
Property 7	If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and not containing any poles.
Property 8	The ROC must be a connected region.

3.10.2 The inverse z-transform

Following our main processing idea (i.e. mapping the discrete time signal into frequency domain, then manipulating the algebraic expressions and afterwards, transferring the results back to the time domain), it is of crucial importance to find the z-transform and its inverse for given discrete-time signals and linear systems. In order to determine the inverse z-transform from a given algebraic expression and associated ROC, recognizing certain transform pairs, known as "inspection method" is the first way. However, sometimes $X(z)$ may not be given explicitly in an available table. In this case, it may be possible to reformulate the expression for $X(z)$ as a sum of simpler terms, each of which exists in the table. This is the case for any rational function, since we can obtain a partial fraction expansion and easily identify the sequences corresponding to the individual terms. To see how to obtain a partial fraction expansion, let us assume that $X(z)$ is expressed as a ratio of polynomials in z^{-1}; i.e.,

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}. \tag{3-73}$$

Such z-transforms arise frequently in the study of linear time-invariant systems. An equivalent expression is in the form

$$X(z) = \frac{b_0 \prod_{k=1}^{M}(1-c_k z^{-1})}{a_0 \prod_{k=1}^{n}(1-d_k z^{-1})}. \tag{3-74}$$

where c_k's and d_k's are respectively, non-zero zeroes and poles of X(z). In case that $M < N$ and the poles are first order, $X(z)$ can be expressed as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{(1-d_k z^{-1})}. \tag{3-75}$$

where:

$$A_k = (1 - d_k z^{-1})X(z)|_{z=d_k}. \tag{3-76}$$

Example 3-13: Second order z-transform

Consider for a given sequence $x[n]$, the z-transform is

$$X(z) = \frac{1}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})} \qquad , |z| > \frac{1}{2} \tag{3-77}$$

the poles and zeroes of $X(z)$ is depicted in Figure 3-15. Since the poles are first order, $X(z)$ can be expressed as

$$X(z) = \frac{A_1}{(1-\frac{1}{4}z^{-1})} + \frac{A_2}{(1-\frac{1}{2}z^{-1})}. \quad (3\text{-}78)$$

From Eq. (3-76),

$$A_1 = \left(1 - \frac{1}{4}z^{-1}\right) X(z)|_{z=\frac{1}{4}} = -1$$
$$A_2 = \left(1 - \frac{1}{2}z^{-1}\right) X(z)|_{z=\frac{1}{2}} = 2 \quad ,$$

therefore,

$$X(z) = \frac{-1}{(1-\frac{1}{4}z^{-1})} + \frac{2}{(1-\frac{1}{2}z^{-1})}.$$

From the ROC and the property 5 in Table 3-4, $x[n]$ is right-hand side; hence, from Figure 3-14 and linearity of z-transform, the inverse transform would be:

$$x[n] = -(\frac{1}{4})^n u[n] + 2\,(\frac{1}{2})^n u[n]$$

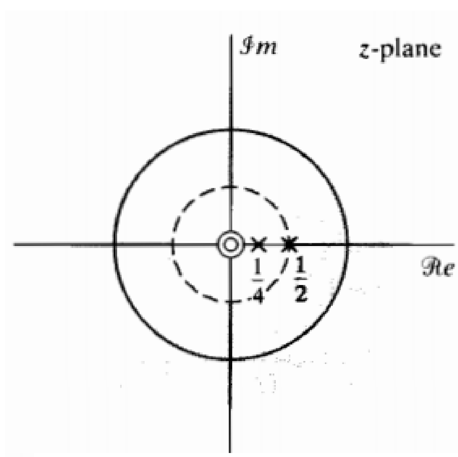

Figure 3-15: zeroes and poles of $X(z)$ in Example 3-13 [1].

3.11 z- Transform Properties

Similar to Fourier transform, z- transform has some properties useful for practical applications. Table 3-5 summarizes these properties.

Table 3-5 Z-transform theorems.

		sequences $x[n]$, $y[n]$	z- transform $X(z)$ with $ROC = R_x, Y(z)$ with $ROC = R_y$
1	Linearity	$ax[n] + by[n]$	$aX(z) + bY(z), ROC = R_x \cap R_y$
2	Time Shifting	$x[n - n_d], n_d$ integer	$z^{-n_d}X(z), ROC = R_x$ (except for probably addition or deletion $z = 0$ or $z = \infty$)
3	Frequency Shifting	$z_0^n x[n]$	$X(z/z_0)$, $ROC = \|z_0\|R_x$
4	Complex Conjugate	$x^*[n]$	$X^*(z^*), ROC = R_x$
		$Re\{x[n]\}$	$\frac{1}{2}(X(z) + X^*(z^*))$, ROC contains R_x
		$Im\{x[n]\}$	$\frac{1}{2j}(X(z) - X^*(z^*))$, ROC contains R_x
5	Time Reversal	$x^*[-n]$	$X^*(1/z^*), ROC = {}^1/_{R_x}$
6	Differentiation in Frequency	$nx[n]$	$-z\frac{dX(z)}{dz}, ROC = R_x$
7	Convolution Theorem	$x[n] * y[n]$	$X(z)Y(z)$, $ROC = R_x \cap R_y$
8	Initial Value Theorem	$x[n] = 0$ for $n < 0$	$\lim_{z\to\infty} X(z) = x[0]$

Example 3-14: Convolution using z-transform

Given $x[n] = a^n u[n]$ and $h[n] = u[n]$, we want to determine $y[n] = x[n] * h[n]$ using z-transform. The corresponding z-transforms are:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}, \quad |z| > |a|,$$

and

$$H(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}, \quad |z| > 1.$$

then, if $|a| < 1$, the z-transform of the convolution of $x[n]$ and $h[n]$ is:

$$Y(z) = X(z)H(z) = \left(\frac{1}{1-az^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right) = \frac{z^2}{(z-a)(z-1)}, \quad |z| > 1.$$

Using Eq. (3-76) and in a partial fraction expression, we obtain:

$$Y(z) = \frac{1}{(1-a)}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right) \quad |z| > 1.$$

then, the sequence $y[n]$ can be obtained by the inverse z-transform of $Y(z)$:

$$y[n] = \frac{1}{(1-a)}\left(u[n] - a^{n+1}u[n]\right).$$

3.12 Discrete- Time Random Signals

In previous sections, we have seen the time-domain and the frequency-domain representations of discrete-time signals and systems. However, until now, we have assumed that the signals are deterministic and each value of a sequence is uniquely available by a mathematical expression or a table of data.

In many situations, the processes that generate signals are so complex and it is extremely difficult (or even impossible) to describe the signal by a mathematical expression. In such cases, the signal is usually modeled as a stochastic process.

As stated in Chapter 1, a stochastic signal is a member of an ensemble of discrete-time signals that is characterized by a set of probability density functions and statistical properties. More specifically, each individual sample $x[n]$ of a particular random signal is assumed to be an outcome of some underlying random variable, $\mathbf{x}_n$. A collection of such *random variables*, one for each sample time, $-\infty < n < \infty$ represents the entire signal. This collection of random variables is called a *random process*. A random process is described by individual and joint probability distributions of all the random variables. Their description in terms of averages then can be computed from assumed probability laws or may be estimated from specific signals.

As a matter of fact, stochastic signals are not absolutely summable or square summable; therefore, it is not possible to directly calculate Fourier transforms for them. However, many of the properties of such signals can be summarized in terms of averages such as the *autocorrelation* or *autocovariance* sequence, for which the Fourier transform often exists. In particular, the effect of processing stochastic signals with a discrete-time linear system can be conveniently described in terms of the effect of the system on the autocovariance sequence.

To continue reading this discussion, the reader needs to be familiar with basic concepts of stochastic processes, such as averages, correlation and covariance functions. This lecture notes for PSM course provides more details on these concepts.

As introduced in Chapter 1, one class of stochastic processes are stationary random signals for which the statistical properties don't change over the time. For simplicity, we limit our focus on this signals and their representation in the context of processing with linear time-invariant systems.

Let's assume $x[n]$ is a real-valued sequence that is a sample sequence of a stationary discrete-time random process and h[n] is the real impulse response of a given stable linear time-invariant system. The output of the system would also be a sample function of a random process related to the input process by the linear transformation where $y[n] = x[n] * h[n]$.

Dealing with a stationary random signal, it is more convenient to characterize it by its mean, m_x, its variance, σ_x^2 and its autocorrelation function, $\varphi_{xx}[m]$. We wish to find the similar information for the output random process, $y[n]$. Therefore, we need to derive input-output relationships for these quantities. The means of the input and output processes are, respectively,

$$m_{x_n} = \mathcal{E}\{\mathbf{x}_n\} \ , \ m_{y_n} = \mathcal{E}\{\mathbf{y}_n\} \tag{3-79}$$

where $\mathcal{E}\{.\}$ stands for expected value. In this course, we don't need to distinguish the random variables $\mathbf{x}_n$ and $\mathbf{y}_n$, and their specific values, $x[n]$ and $y[n]$ such that:

$$m_x[n] = \mathcal{E}\{x[n]\} \ , \ m_y[n] = \mathcal{E}\{y[n]\}. \tag{3-80}$$

If $x[n]$ and y[n] are stationary, then $m_x[n]$ and $m_y[n]$ are independent of n and will be written as m_x and m_y.

Since $y[n] = x[n] * h[n]$ and regarding the fact that the expected value of a sum is the sum of the expected values, the mean of the output process is

$$m_y[n] = \mathcal{E}\{y[n]\} = \sum_{k=-\infty}^{\infty} h[k]\mathcal{E}\{x[n-k]\}, \tag{3-81}$$

As the input is stationary, $m_x[n- \ k] = m_x$, and consequently,

$$m_y[n] = m_x \sum_{k=-\infty}^{\infty} h[k]. \tag{3-82}$$

From Eq. (3-82), it is obvious that the mean of the output is also constant. Following Eq. (3-34) an equivalent expression to Eq. (3-82) in terms of the frequency response is

$$m_y = H(e^{j0})m_x. \tag{3-83}$$

Back to $y[n] = x[n] * h[n]$, this time we want to find the autocorrelation function of the output process:

$$\varphi_{yy}[n,n+m] = \mathcal{E}\{y[n]y[n+m]\} \tag{3-84}$$
$$= \mathcal{E}\{\sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} h[k]h[r]x[n-k]x[n+m-r]\}$$
$$= \sum_{k=-\infty}^{\infty} h[k]\sum_{r=-\infty}^{\infty} h[r]\,\mathcal{E}\{x[n-k]x[n+m-r]\}$$

Since $x[n]$ is assumed to be stationary, $\mathcal{E}\{x[n-k]x[n+m-r]\}$ depends only on the time difference, i.e., $m + k - r$. Therefore,

$$\varphi_{yy}[n,n+m] = \sum_{k=-\infty}^{\infty} h[k]\sum_{r=-\infty}^{\infty} h[r]\,\varphi_{xx}[m+k-r] = \varphi_{yy}[m] \tag{3-85}$$

Therefore, the output autocorrelation sequence also depends only on the time difference m. Thus, for a linear time-invariant system having stationary input, the output is also stationary.

By making the substitution $l = r - k$ and defining the autocorrelation sequence of $h[n]$ as $c_{hh}[l] = \sum_{k=-\infty}^{\infty} h[k]h[l+k]$, we can express Eq. (3-85) as

$$\varphi_{yy}[m] = \sum_{l=-\infty}^{\infty} \varphi_{xx}[m-l]\sum_{k=-\infty}^{\infty} h[k]h[l+k] = \sum_{l=-\infty}^{\infty} \varphi_{xx}[m-l]c_{hh}[l] \tag{3-86}$$

It is worth mentioning that $c_{hh}[l]$ is simply the discrete convolution of $h[n]$ with $h[-n]$.

Eq. (3-86) means that the autocorrelation of the output of a linear system is the convolution of the autocorrelation of the input with the aperiodic autocorrelation of the system impulse response.

Keeping the convolution property of Fourier transform, Eq. (3-86) suggests that the response of a linear time-invariant system to a stochastic input can also be characterized in Frequency domain. Assume, for convenience, that $m_x = 0$ such that the autocorrelation and autocovariance sequences are identical. Then, with $\Phi_{xx}(e^{j\omega})$, $\Phi_{yy}(e^{j\omega})$, and $C_{hh}(e^{j\omega})$ denoting the Fourier transforms of $\varphi_{xx}[m]$, $\varphi_{yy}[m]$, and $c_{hh}[l]$, respectively, from Eq. (3-86),

$$\Phi_{yy}(e^{j\omega}) = \Phi_{xx}(e^{j\omega})C_{hh}(e^{j\omega}) \tag{3-87}$$

Also, from the definition of $c_{hh}[l]$,

$$C_{hh}(e^{j\omega}) = \mathrm{H}(e^{j\omega})H^*(e^{j\omega}) = |\mathrm{H}(e^{j\omega})|^2 , \tag{3-88}$$

So,

$$\Phi_{yy}(e^{j\omega}) = |\mathrm{H}(e^{j\omega})|^2\Phi_{xx}(e^{j\omega}) \tag{3-89}$$

Eq. (3-89) defines the term *power spectrum density* of the output signal. Specifically, regarding the Parseval's theorem (see Table 3-3), the total average power of the output is defined as

$$\mathcal{E}\{y^2[n]\} = \varphi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{yy}(e^{j\omega})\, d\omega. \tag{3-90}$$

Now, substituting the Eq. (3-90) in Eq. (3-89), we have

$$\mathcal{E}\{y^2[n]\} = \varphi_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\mathrm{H}(e^{j\omega})\right|^2 \Phi_{xx}(e^{j\omega})\, d\omega. \tag{3-91}$$

Example 3-15: White Noise

A white-noise signal is a signal for which $\varphi_{xx}[m] = \sigma_x^2\delta[m]$. If the signal has zero mean, then the power spectrum of a white noise signal is a constant, i.e.,

$$\Phi_{xx}(e^{j\omega}) = \sigma_x^2 \qquad \text{for all } \omega.$$

The average power of a white-noise signal is therefore

$$\varphi_{xx}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{xx}(e^{j\omega})\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sigma_x^2\, d\omega = \sigma_x^2.$$

The concept of white noise is also useful in the representation of random signals whose power spectra are not constant with frequency. For example, a random signal $y[n]$ with power spectrum $\Phi_{yy}(e^{j\omega})$ can be assumed to be the output of a LTI system with a white-noise input. In other words, we use Eq. (3-89) to define a system with frequency response $\mathrm{H}(e^{j\omega})$ to satisfy the equation $\Phi_{yy}(e^{j\omega}) = \left|\mathrm{H}(e^{j\omega})\right|^2\sigma_x^2$, where σ_x^2 is the average power of the assumed white-noise input signal. We wish to set the average power of this input signal to give the correct average power for $y[n]$. For instance, let's assume $h[n] = a^n u[n]$ and therefore, $\mathrm{H}(e^{j\omega}) = \frac{1}{1-ae^{j\omega}}$. Then we can represent all random signals whose power spectrum is in the form of

$$\Phi_{yy}(e^{j\omega}) = \left|\frac{1}{1-ae^{j\omega}}\right|^2 \sigma_x^2 = \frac{\sigma_x^2}{1+a^2-2a\cos\omega}.$$

In this context, the last property of the LTI system in which we are interested is the cross-correlation between the input and output of a linear time-invariant system:

$$\varphi_{xy}[m] = \mathcal{E}\{x[n]y[n+m]\} \tag{3-92}$$

$$= \mathcal{E}\{x[n]\textstyle\sum_{k=-\infty}^{\infty} h[k]\, x[n+m-k]\}$$

$$= \sum_{k=-\infty}^{\infty} h[k]\varphi_{xx}[m-k]$$

It means that the cross-correlation between input and output is the convolution of the impulse response with the input autocorrelation sequence.

The Fourier transform of Eq. (3-92) is

$$\Phi_{xy}(e^{j\omega}) = \mathrm{H}(e^{j\omega})\Phi_{xx}(e^{j\omega}). \tag{3-93}$$