Exercises for FLL, Fall 2017, sheet 3 – Solutions

Return Thu Oct 19, in class

Exercise 1. Is the language $L = \{w \in \{0, 1\}^* \mid w$ contains an equal number of 0's and 1's$\}$ regular? Prove your answer.

Solution. Language is not regular. Proof via PL. Assume it is regular and PL holds. Let n be PL constant. Then $w = 0^n1^n \in L$. But according to PL, some nonempty subword $y = 0^m$ of the first part exists such that also $0^{n+m}1^n \in L$, which it isn't , contradiction, thus L no regular.s

Exercise 1a [a little more tricky, optional]. Prove that the language $L = \{0^n \mid n = pq$ for two primes $p, q\}$ is not regular.

Solution. Assume L is regular. Then by the pumping lemma, there exists a constant c such that, if $w \in L$, $|w| > c$, w can be written as uvx, $r = |v| > 0$, such that $uv^ix \in L$ for all $i \geq 0$. Let p, q such that $n = pq > c$. Then $0^{pq} \in L$ and by the pumping lemma, $0^{pq+ir} \in L$ for all $i \geq 0$. Specifically, for $i = pq$ we obtain $0^{pq(1+r)} \in L$. But $pq(1+r)$ is not the product of two primes, so by the definition of L, $0^{pq(1+r)} \notin L$. Contradiction, therefore the assumption that L is regular is wrong, therefore L is not regular.

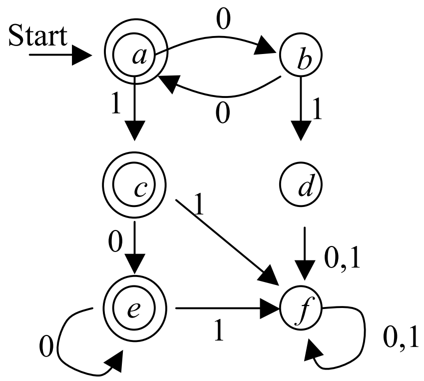
Exercise 2. Prove or disprove the following conjecture:

Let M be some regular language over $\Sigma = \{0,1\}$. Define $L_{|M|} = \{0^n \in \{0\}^* \mid n = |v|$ for some word $v \in M\}$. Then $L_{|M|}$ is regular.

Note: this is easy to solve using the tool of language homomorphisms. It is also possible to solve this problem without homomorphisms, also not difficult.

Solution. The conjecture is true. Proof with homomorphisms: Consider the (unique!) homomorphism h: $\{0,1\} \to \{0\}$. Then $L_{|M|} = h(M)$ and by Proposition 3.11, $L_{|M|}$ is regular. Proof without homomorphims: Let A be a DFA for M. Relabel all "1"-transitions in A by 0, obtain an NFA A'. This NFA accepts $L_{|M|}$. To see this, first consider some $v \in M$. It has a path in A that ends in an accepting state; the same path in A' leads the input word $0^{|v|}$ to the same accepting state. This if $n = |v|$ for some word $v \in M$, $0^n \in L(A')$, that is, $L_{|M|} \subseteq L(A')$. Conversely, let $0^n \in L(A')$. Then there is a path of 0-transitions of length n in A' that leads from the starting state to some accepting state. If you replace the 0's on this path that were obtained from 1 -> 0 relabelings by the original 1's, connecting the $\{0, 1\}$ transitions on this path you get a word $v \in M$, $n = |v|$. Hence $L(A') \subseteq L_{|M|}$. Therefore, $L(A') = L_{|M|}$ and $L_{|M|}$ is regular.

Exercise 3. Minimize the DFA shown in the figure by using the table filling method. Deliverables: the filling table, the set of states of the minimal DFA, and a graph representation of the minimal DFA.

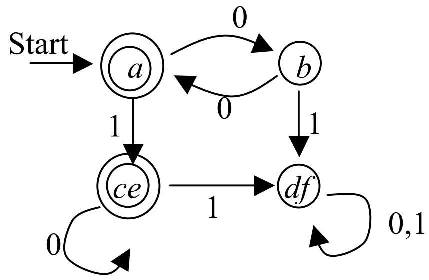


Solution. Manual labour by accurate following of the recipe...

Table:

b	x_1				
c	x_2	x_1			
d	x_1	x_2	x_1		
e	x_2	x_1		x_1	
f	x_1	x_2	x_1		x_1
	a	*b*	*c*	*d*	*e*

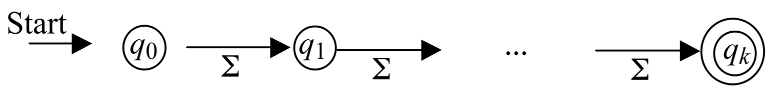
Minimal DFA:



New states: $\{a\}$, $\{b\}$, $\{c,d\}$, $\{e,f\}$

Exercise 4. Let L be a regular language specified by a DFA, NFA, ε-NFA, or regexp. Show that it is decidable whether $L = \Sigma^k$ for some $k > 0$.

Solution. There are many ways of how this can be decided. One elegant way is to first construct the minimal DFA A for L. Then obviously $L = \Sigma^k$ for some k iff A has the form



Note. The use of the word "obviously" in mathematical proofs is a delicate affair. One never knows what the reader is ready to accept as obvious. Here I think we have a borderline case, and one might feel the need to prove that if $L = \Sigma^k$ then the minimal automaton actually has the given form (it is really obviously obvious that this kind of DFA accepts $L = \Sigma^k$, so the only possibly questionable claim is its minimality). Minimality of DFAs of the shown form could be proven by going through the table-filling algorithm and showing that all the shown states are distinguishable. This would be a case for extra grading points.