Exercises for FFL, Fall 2017, sheet 8

Return Thursday Nov 23, in class.

Exercise 1. Let $S = \{c, f\}$ be a signature consisting of a constant symbol c and a binary function symbol f. Let $\mathcal{A} = (A, c^A, f^A)$ be the S-structure over carrier $A = \{1, 2\}$ with $c^A = 1$ and f^A given by $f^A(1, 1) = 1, f^A(1, 2) = 2, f^A(2, 1) = f^A(2, 2) = 1$. Your task: represent $\mathcal{A}$ as a set, coding lists by iterated nested pairs (right-associative) and representing these (nested) pairs as sets (so ultimately there will be only "{}" symbols, no more "()" symbols). Using a text processor's copy-paste capabilities saves a lot of time in this task, prevents copy errors, and makes it possible for the TAs to read your result, so please use a text processor for this task.

Solution. Let us work from the inside to the outside. The graph set of f^A, written using pairs, is $\{((1,1),1), ((1,2),2), ((2,1),1), ((2,2),2)\}$. Turning the argument pairs into sets turns this to $\{(\{1,\{1\}\},1), (\{1,\{1,2\}\},2), (\{2,\{1,2\}\},1), (\{2,\{2\}\},2)\}$ and the outer pair brackets into sets to $f^A = \{\{\{1,\{1\}\},\{\{1,\{1\}\},1\}\}, \{\{1,\{1,2\}\},\{\{1,\{1,2\}\},2\}\}, \{\{2,\{1,2\}\},\{\{2,\{1,2\}\},1\}\}, \{\{2,\{2\}\},\{\{2,\{2\}\},2\}\}\}$.

c^A is 1.

The pair (c^A, f^A) is $\{c^A, \{ c^A, f^A\}\} = \{1, \{1, \{\{\{1,\{1\}\},\{\{1,\{1\}\},1\}\}, \{\{1,\{1,2\}\},\{\{1,\{1,2\}\},2\}\}, \{\{2,\{1,2\}\},\{\{2,\{1,2\}\},1\}\}, \{\{2,\{2\}\},\{\{2,\{2\}\},2\}\}\}\}\}$.

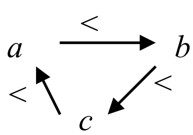
The triple (A, c^A, f^A) then becomes $(A, (c^A, f^A)) = \{A, \{A, (c^A, f^A)\}\} = \{\{1,2\}, \{\{1,2\}, \{1, \{1, \{\{\{1,\{1\}\},\{\{1,\{1\}\},1\}\}, \{\{1,\{1,2\}\},\{\{1,\{1,2\}\},2\}\}, \{\{2,\{1,2\}\},\{\{2,\{1,2\}\},1\}\}, \{\{2,\{2\}\},\{\{2,\{2\}\},2\}\}\}\}\}\}\}$.

Exercise 2.
Let $S = \{<\}$, where $<$ is a binary relation symbol. Characterize in words the class of all S-structures $\mathcal{A}$ which are models of

$$\varphi = \forall x_1 \forall x_2 \forall x_3 (((((\neg\, x_1 = x_2 \wedge \neg\, x_2 = x_3) \wedge \neg\, x_1 = x_3) \wedge <x_1\, x_2) \wedge <x_2\, x_3) \rightarrow \neg <x_3\, x_1).$$

Furthermore, give two concrete S-structures, one of which is a model of φ and the other isn't. Present your structures by specifying the carrier and the extension of $<^A$.

Solution. The models of φ are exactly those $\{<\}$-structures that contain no <-cycle of length 3. The simplest S-structure that is a model of φ is given by a singleton set A and empty <, that is, $A = \{a\}$ and $<^A = \emptyset$ (graph-like representation: a single point). The simplest S-structure that is not a model of φ is an isolated 3-cycle of <, that is, $A = \{a, b, c\}$ and $<^A = \{(a,b), (b,c), (c,a)\}$:



Exercise 3. Let S = {father-of, Tom, Anne}, where father-of is a unary function symbol and Tom, Anne are constant symbols. Give a concrete S-structure which is *not* a model of the S-

expression "father-of Anne = Tom". And give a concrete *S*-structure which *is* a model of the *S*-expression "father-of Tom = Anne".

Solution (out of a zillion possibilities).
(a) $\mathcal{A} = (A, \text{Tom}^{\mathcal{A}}, \text{Anne}^{\mathcal{A}}, \text{father-of}^{\mathcal{A}}) = (\{1, 2\}, 1, 2, \{(1, 1), (2, 2)\})$
(b) $\mathcal{A} = (A, \text{Tom}^{\mathcal{A}}, \text{Anne}^{\mathcal{A}}, \text{father-of}^{\mathcal{A}}) = (\{1\}, 1, 1, \{(1, 1)\})$

Exercise 4. You know that a binary relation is called an *equivalence relation* if it is (i) reflexive, (ii) symmetric, and (iii) transitive. Let $S = \{\equiv\}$, where $\equiv$ is a binary relation symbol. Give an *S*-expression φ such that in any *S*-structure $\mathcal{A} = (A, \equiv^{\mathcal{A}})$, which is a model of φ, $\equiv^{\mathcal{A}}$ is an equivalence relation on *A*.

Solution. The following *S*-expression characterizes $\equiv^{\mathcal{A}}$ to be an equivalence relation:

$$\forall x (\equiv x\, x) \land \forall x \forall y (\equiv x\, y \to \equiv y\, x) \land \forall x \forall y \forall z ((\equiv x\, y \land \equiv y\, z) \to \equiv x\, z)$$