Machine Learning, Spring 2018: Exercise Sheet 6

It's paper and pencil time again.

Problem 1 (linear algebra training). Let $x_1, ..., x_m \in \mathbb{R}^n$ be m linearly independent n-dimensional vectors, and let μ be their mean. Prove that the centered points $\bar{x}_1,\ldots,\bar{x}_m = x_1 - \mu,\ldots,x_m - \mu$ span an m–1 dimensional subspace of $\mathbb{R}^n$. (Recall that a set $x_1, ..., x_m$ of vectors is called linearly independent if $a_1 x_1 + ... + a_m x_m = \mathbf{0}$ implies $a_1 = ... = a_m = 0$.)

Solution. First we show that $\bar{x}_1,\ldots,\bar{x}_m$ are linearly dependent. Using unit combination coefficients $a_i = 1$ for all $1 \le i \le m$, we find that

$$\bar{x}_1 + \ldots + \bar{x}_m = (x_1 - \mu) + \ldots + (x_m - \mu) =$$

$$\sum_{i=1}^m x_i - m\mu = \sum_{i=1}^m x_i - m\frac{1}{m}\sum_{i=1}^m x_i = 0,$$

hence $\bar{x}_1,\ldots,\bar{x}_m$ are linearly dependent and thus span a subspace of dimension less than m.

Next we show that $\bar{x}_1,\ldots,\bar{x}_{m-1}$ are linearly independent (then we are done, because then $\bar{x}_1,\ldots,\bar{x}_m$ span an m–1 dimensional subspace). Consider a linear combination satisfying $a_1\bar{x}_1 + \ldots + a_{m-1}\bar{x}_m = 0$. We show that this implies that all a_j are zero:

$$\begin{aligned}
0 &= a_1\bar{x}_1 + \ldots + a_{m-1}\bar{x}_{m-1} \\
&= a_1(x_1 - \mu) + \ldots + a_{m-1}(x_{m-1} - \mu) \\
&= \sum_{j=1}^{m-1} a_j x_j - \left(\sum_{j=1}^{m-1} a_j\right)\mu \\
&= \sum_{j=1}^{m-1} a_j x_j - \left(\sum_{j=1}^{m-1} a_j\right)\frac{1}{m}\sum_{i=1}^m x_i \\
&= -\frac{1}{m}\left(\sum_{j=1}^{m-1} a_j\right)x_m + \sum_{j=1}^{m-1}\left(a_j - \frac{1}{m}\left(\sum_{j=1}^{m-1} a_j\right)\right)x_j
\end{aligned}$$

$$\Rightarrow -\frac{1}{m}\left(\sum_{j=1}^{m-1} a_j\right) = 0 \quad \text{because } x_1,\ldots,x_m \text{ are linearly independent, and}$$

$$\left(a_j - \frac{1}{m}\left(\sum_{j=1}^{m-1} a_j\right)\right) = 0 \quad \text{for all } j$$

$$\Rightarrow a_j = 0 \quad \text{for all } j.$$