PSM Fall 2015, Exercise Sheet 4

Return on Monday Oct 05 in class

Note. You are encouraged to work in teams of two – but no larger. If you work in a team, submit only a single sheet with both names indicated on it. Nicely type-set solutions are highly appreciated.

Problem 1 (20 points) Let Y_1, Y_2, Y_3 be three (real-valued) random variables. Verify that

$$\begin{aligned} VAR[aY_1 + bY_2 + cY_3] &= a^2VAR[Y_1] + b^2VAR[Y_2] + c^2VAR[Y_3] \\ &\quad +2abCOV[Y_1, Y_2] + 2acCOV[Y_1, Y_3] \\ &\quad +2bcCOV[Y_2, Y_3]. \end{aligned}$$

Problem 2 (20 points) The cumulative distribution function for the random variable X is given by

$$F(x) = \begin{cases} 0, & \text{for } x \leq 0, \\ \frac{x}{8}, & \text{for } 0 \leq x \leq 2, \\ \frac{x^2}{16}, & \text{for } 2 \leq x \leq 4, \\ 1, & \text{for } 4 \leq x. \end{cases}$$

a. Find and graph $f(x)$.

b. Find $E[X]$ and $STD[X]$.

c. Simulate1000 values for X and find the sample average and sample standard deviation. Compare these values to part b). For the simulation use the following algorithm (*cumulative distribution function method*):

- Obtain a random number u with $0 \leq u \leq 1$.
- Set $u = F(x)$ for the given CDF F of X.
- Solve for x
- Repeat until you have the required number of simulations.

Problem 3 (15 points)

(6 points)

Let U_1 and U_2 be two independent and uniformly distributed random variables (over the interval [0,1]). Use the rule of substitution (aka transformation rule) or the convolution theorem to determine the probability density functions of the following variables:

a. $Z = 2 \cdot U_1$

b. $Z = U_2^2$

c. $Z = U_1 + U_2$

Problem 4 (15 points) Let X and Y be continuous random variables with joint probability density function

$$f(x, y) = \begin{cases} k, & \text{for } 0 \leq x \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

a. Sketch the region for which $f(x, y) > 0$.

b. Find k?

c. Find $P(X < 0.5, Y > 0.5)$.

d. Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$.

e. Are X and Y independent random variables?

Problem 5 (25 points) Let X_1 be the time a customer waits at a smartphone repair shop for service to begin on her smartphone. Let X_2 be the time it takes to complete the service. Assume X_1 and X_2 are independently distributed as

$$f_{X_1}(x_1) = \begin{cases} 2x_1, & \text{for } 0 \leq x_1 \leq 1 \text{ hour} \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{X_2}(x_2) = \begin{cases} \frac{2}{9}(3 - x_2) & \text{for } 0 \leq x_2 \leq 3 \text{ hours} \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_1 = X_1 + X_2$ be the total time spent at the repair shop and $Y_2 = X_2 \times 30€$ be the cost of labour.

a. Simulate 5000 values of X_1 and X_2 using the *cumulative distribution function method*. For each simulated pair, form Y_1 and Y_2. Estimate the mean and standard deviation of Y_1 and of Y_2 from these simulated scores.

b. Find the joint PDF $g(y_1, y_2), E[Y_1], E[Y_2], , VAR[Y_1]$, and $VAR[Y_2]$. Compare this PDF and the expected values and variances to the simulated values from part a).

Problem 6 (5 points)

The temperature X (Celsius) at a randomly selected point in a commercial refrigerator is a random variable with PDF

$$f(x) = \begin{cases} \frac{x^2}{9}, & \text{for } 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

a. Find $E[X]$ and $STD[X]$.

b. Let Y be the temperature in Fahrenheit. Find $E[Y]$ and $STD[Y]$.