PSM Fall 2015, Exercise Sheet 6

Return on Tuesday Oct 27 in class

Note. You are encouraged to work in teams of two – but no larger. If you work in a team, submit only a single sheet with both names indicated on it. Nicely type-set solutions are highly appreciated.

Problem 1 (100 points total) Let X be a real-valued normal random variable with variance 1 and unknown mean θ, which is to be estimated. Since the experimenter feels that the loss is roughly like squared error $(d-\theta)^2$ when the true θ is small but is like squared relative error $(\frac{d}{\theta}-1)^2$ when θ is larger, he or she chooses loss function $\frac{(\theta-d)^2}{1+\theta^2}$ to reflect this behavior.

a. (5 points) Formally specify the sample space S, the statistical model $\mathcal{P}$, the decision space D and the loss function L.

b. (20 points) Determine and plot on the same graph the risk functions of the six procedures $t_1, \ldots, t_6$ that are defined as follows:

$$\begin{aligned} t_1 &= x \\ t_2 &= \frac{1+x}{2} \\ t_3 &= \frac{x}{2} \\ t_4 &= 2x \\ t_5 &= 0 \\ t_6 &= 1. \end{aligned}$$

[Hint: You can save calculations if you first compute the risk function for a general procedure of the form $t(x) = a + bx$.] [A check: $r_{t_4}(\theta) = \frac{(\theta^2+4)}{(1+\theta^2)}$.]

c. (5 points) From these calculations can you assert that any of these six procedures is inadmissable? Which ones?

d. (10 points) On the basis of the risk functions if any of these six procedures must be used which procedure would you use and why?

e. (20 points) Suppose X is replaced by the vector $(X_1, \ldots, X_n)$ of i.i.d. normal $N(\theta, 1)$ random variables. The procedures corresponding to t_1, t_2, t_3 and t_6 are

$$\begin{aligned} t_{1,n} &= \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}_n \\ t_{2,n} &= \frac{\bar{x}_n + \frac{1}{n}}{1+\frac{1}{n}} \\ t_{3,n} &= \frac{n^{\frac{1}{2}}\bar{x}_n}{(1+n^{\frac{1}{2}})} \\ t_{6,n} &= 1. \end{aligned}$$

Compute the risk functions of these four procedures and plot $n \cdot r_{t_{i,n}}$ for these four procedures. [Hint: Note that $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ follows a

$N(\theta, \frac{1}{n})$-distribution when X_i follows a $N(\theta, 1)$- distribution. Again, you may find it easier to calculate the risk function for a general procedure of the type $t(x_1, \ldots, x_n) = a + b\bar{x}_n$.]

f. (10 points) If n is large, which of the four procedures of part e) would you use, and why? [Your answer to this question might differ from question d) for the case $n = 1$. Does it?]

g. (15 points) Suppose the statistician decides to restrict considerations to procedures $t_{a,b,n} = a + b\bar{x}_n$ of the form mentioned in the hints to question e). He or she is concerned about the behaviour of the risk function when θ is large. Show that the risk function approaches 0 as $\mid \theta \mid \rightarrow \infty$ if and only if $b = 1$; and that *among procedures* with $b = 1$, the choice $a = 0$ gives uniformly smallest risk function.

h. (15 points) Show that $t_{6,n}$ is admissible for all n. [Hint: If t' is better than $t_{6,n}$ what can you conclude from $r_{t'}(1) \leq r_{t_{6,n}}(1)$?]