PSM Fall 2015, Exercise Sheet 7

Return on Tuesday Nov 03 in class

Note. You are encouraged to work in teams of two – but no larger. If you work in a team, submit only a single sheet with both names indicated on it. Nicely type-set solutions are highly appreciated.

Problem 1 (20 points total) Let $(X_1, \ldots, X_n)$ be i.i.d. RVs with finite but unknown mean μ and variance σ^2.

a. Show that $t_1(x_1, \ldots, x_n) = \bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$ is an unbiased estimator for the unknown mean $\mu \in \mathbb{R}$.

b. Compute the bias of $t_2(x_1, \ldots, x_n) = S_n^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$ estimating the unknown variance σ^2.

c. Show that $t_3(x_1, \ldots, x_n) = S^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2$ is an unbiased estimator for the unknown variance σ^2.

d. Assuming μ is known show that $t_4(x_1, \ldots, x_n) = S_{n,\mu}^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \mu)^2$ is an unbiased estimator for the unknown variance σ^2.

Problem 1 (40 points total) Let $(X_1, \ldots, X_n)$ be i.i.d. normal $N(\theta, 1)$ random variables with variance 1 and unknown mean θ, which is to be estimated. Since the experimenter feels that the loss is roughly like squared error $(d-\theta)^2$ when the true θ is small but is like squared relative error $(\frac{d}{\theta} - 1)^2$ when θ is larger, he or she chooses loss function $\frac{(\theta-d)^2}{1+\theta^2}$ to reflect this behavior.

a. (20 points) . Show that procedure

$$t_{2,n} = \frac{\bar{x}_n + \frac{1}{n}}{1 + \frac{1}{n}}$$

is Bayes relatively to the prior density

$$f_{\xi_1}(\theta) = C_2\phi(\theta - 1)(1 + \theta^2),$$

where ϕ denotes the density function of the standard normal distribution $N(0, 1)$ and C_2 is a suitable constant. [You need not determine C_2. It suffices to verify that the given functions of θ have finite integrals, so that one knows that such a C_2 exists.]

b. (20 points) . Show that procedure

$$t_{3,n} = \frac{n^{\frac{1}{2}}\bar{x}_n}{(1 + n^{\frac{1}{2}})}$$

is Bayes relatively to the prior density

$$f_{\xi_1}(\theta) = C_3\phi(\theta)(1 + \theta^2),$$

where ϕ denotes the density function of the standard normal distribution $N(0, 1)$ and C_3 is a suitable constant. [You need not determine C_3. It suffices to verify that the given functions of θ have finite integrals, so that one knows that such a C_3 exists.]

Problem 2 (40 points total) Suppose $X = (X_1, X_2)$ where the X_i's are i.i.d., each with probability density function

$$f_\theta(x) = \begin{cases} \frac{3x^2}{\theta^3} & \text{if } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{P} = \{\theta : \theta > 0\}$, $D = \{d : d > 0\}$, and $L(\theta, d) = (\theta - d)^2$.

a. (10 points) Show that each of the following two estimators is an unbiased estimator:

$$\begin{aligned} t(x_1, x_2) &= \frac{2}{3}(x_1 + x_2) \\ t'(x_1, x_2) &= \frac{7}{6}\max(x_1, x_2) \end{aligned}$$

[hint: show that $Y = \max(X_1, X_2)$ has density $6y^5/\theta^6, 0 < y < theta$; you need not compute the density of $X_1 + X_2$ to compute the risk function later.]

b. (15 points) Find the risk function of each of the two procedures t and t' and show that the former is 60% larger than the latter.

c. (10 points) Show that among all procedures of the form $t'_c(x_1, x_2) = c\max(x_1, x_2)$ the procedure $t'_{\frac{8}{7}}$ is uniformly best.

d. (5 points) Check whether procedure $t'_{\frac{8}{7}}$ is unbiased.