PSM Fall 2016, Exercise Sheet 2

Return on Thursday Sep 22 in class

Note. You are encouraged to work in teams of two — but no larger. If you work in a team, submit only a single sheet with both names indicated on it. Nicely type-set solutions are highly appreciated.

Problem 1 (30 pts). (Motto: make friends with σ-fields)

Let $(M, \mathcal{G})$ be a measurable space consisting of a set M with a σ-fields $\mathcal{G} \subseteq \mathrm{Pot}(M)$. That is, $\mathcal{G}$ satisfies the conditions spelled out in Definition 7.2 in the lecture notes. Use these conditions to prove the following facts:

(i) closure under countable intersection: if $A_i \in \mathcal{G}$, $(i \in \mathbb{N})$, then $\bigcap_{i\in\mathbb{N}} A_i \in \mathcal{G}$.

(ii) If $B \subseteq M$ is some non-empty subset of M, then $\mathcal{G}' = \{A \cap B \mid A \in \mathcal{G}\}$ is a σ-field on B. It is called the *restriction* of $\mathcal{G}$ on B.

(iii) If $\mathcal{G}$ and $\mathcal{H}$ are σ-fields on M, then $\mathcal{G} \cap \mathcal{H}$ is a σ-field on M.

Problem 2 (pts) (Make friends with the Borel σ-fields)

The Borel σ-field $\mathcal{B} \subseteq \mathrm{Pot}(\mathbb{R})$ is a particular, widely used, "natural" σ-field on the reals. We will explore it a little. $\mathcal{B}$ contains (among many other subsets of the reals) all the closed nonempty intervals $[a, b] \subseteq \mathbb{R}$, where $a < b$. Use this fact and the definition of σ-fields to show the following facts:

(i) the half-infinite intervals $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$ are elements of $\mathcal{B}$.

(ii) the open intervals (a, b) are elements of $\mathcal{B}$. (Note: the closed interval $[a, b]$ is the set $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ and the open interval is the set $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

(iii) Any countable (finite or infinite) subset of $\mathbb{R}$ is in $\mathcal{B}$.

(iv) Teaser question (optional). Borel σ-fields are also defined on higher-dimensional Euclidean spaces $\mathbb{R}^n$. Similar to the one-dimensional case, the Borel σ-field $\mathcal{B}^n$ on $\mathbb{R}^n$ contains all closed hypercubes $[a_1, b_1] \times \ldots \times [a_n, b_n]$. Use this to show that $\mathcal{B}^2$ on $\mathbb{R}^2$ also contains the diagonal line set $\{(x, x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. (In fact $\mathcal{B}^2$ contains any set of points that can be visualized as a curve in $\mathbb{R}^2$).

Some background information worth knowing:

- Borel σ-fields are in fact *defined* by the condition that they are the smallest σ-fields which contain the intervals (one-dimensional case) or the hypercubes (higher dimensional cases).
- By using the statement (ii) from problem 2, Borel σ-fields can be defined on any subset of $\mathbb{R}^n$, for instance on the unit sphere or on linear subspaces.

Problem 3 (pts) Prove the three facts stated in the lecture notes after Definition 8.1:

(i) $P(\emptyset) = 0$

(ii) $P(F^c) = 1 - P(F)$

(iii) $F \subseteq F' \Rightarrow P(F) \leq P(F')$