PSM Fall 2016, Exercise Sheet 3 – Sample solution

Return on Thursday October 6 in class

Note. You are encouraged to work in teams of two – but no larger. If you work in a team, submit only a single sheet with both names indicated on it. Nicely type-set solutions are highly appreciated.

Problem 1 (10 points) Let $\Omega = \{\omega_1, \dots, \omega_6\}$ be a (uncommonly small) universe. Define two RVs $X, Y : \Omega \to \{\text{small, medium, large, x-large}\}$ which are identically distributed but not identical.

Let P be the uniform distribution on Ω and $X : \Omega \to \{\text{small, medium, large, x-large}\}$ a RV with

$X(\omega_1) = \text{small}, X(\omega_2) = X(\omega_3) = \text{medium}, X(\omega_4) = X(\omega_5) = \text{large}, X(\omega_6) = \text{x-large}.$

$$X(\omega) = \begin{cases} small & \text{for } \omega = \omega_1 \\ medium & \text{for } \omega \in \{\omega_2, \omega_3\} \\ large & \text{for } \omega \in \{\omega_4, \omega_5\} \\ x-large & \text{for } \omega = \omega_6 \end{cases}$$

$$Y(\omega) = \begin{cases} small & \text{for } \omega = \omega_2 \\ medium & \text{for } \omega \in \{\omega_1, \omega_4\} \\ large & \text{for } \omega \in \{\omega_3, \omega_6\} \\ x-large & \text{for } \omega = \omega_5. \end{cases}$$

Then the PMFs for both random variables are identical since

$$\begin{aligned} p_X(s) &= \begin{cases} \frac{1}{6} & \text{for } s \in \{\text{small, x-large}\} \\ \frac{1}{3} & \text{for } s \in \{\text{medium, large}\} \end{cases} \\ &= p_Y(s). \end{aligned}$$

and consequently X and Y are identically distributed.

Obviously, X and Y are not identical, e.g. $X(\omega_1) = \text{small} \neq \text{medium} = Y(omega_1)$.

Problem 2 (15 points) Let $X_1 : \Omega \to \{1, 2, 3, 4, 5\}, X_2 : \Omega \to \{\text{red, green, blue}\}$ and $X_3 : \Omega \to \{fast, slow\}$ be three RVs with values in $S_1 = \{1, 2, 3, 4, 5, \}$, $S_2 = \{\text{red, green, blue}\}$, and $S_3 = \{fast, slow\}$. Invent and fully specify a joint distribution of these three RVs which makes them jointly independent.

Let P_{X_1} be the uniform distribution on $\{1, 2, 3, 4, 5\}$, P_{X_2} be the uniform distribution on $\{\text{red, green, blue}\}$ and P_{X_3} be the Bernoulli distribution with parameter $p = 0.25$ for the event $fast$. To ensure joint independence we take the joint distribution as the product of the marginal PMFs. So, we get

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \begin{cases} \frac{1}{60} & \text{if } x_3 = \text{fast} \\ \frac{1}{20} & \text{if } x_3 = \text{slow}. \end{cases}$$

Problem 3 (15 points) Signals that are either 0's or 1's are sent in a noisy communication circuit. The signal received is the signal sent plus a random variable, ϵ, that is normally distributed with mean $\mu = 0$ and standard deviation $\sigma = \frac{1}{3}$. If a 0 is sent, the receiver will record a 0 if the signal received is at most a value

v; otherwise a 1 is recorded. Determine v such that the probability that a 1 is recorded when a 0 is actually sent is 0.90.

$P(1 \text{ recorded}, 0 \text{ sent}) = P(\epsilon > v) = 1 - \Phi_{0,1/3}(v) = 0.9$ if $\Phi_{0,1/3}(v) = 0.1$. Hence $v = q_{0,1/3}(0.1) = -0.4271839$.

Problem 4 (15 points) Suppose that $X \sim N(\mu, \sigma)$ and let $Y = e^X$.

1. Find the mean and variance of Y.
2. Find the probability density function of Y. The result is called the *log-normal* probability density function because $\log Y \sim N(\mu, \sigma)$.

1. The easiest solution for this question uses the moment generating function for the normal distribution which is $m(t) = E[e^{tX}] = \exp(\mu t + \frac{1}{2}\sigma^2 t^2), t \in \mathbb{R}$. We then straightforwardly get

$$\begin{aligned} E[Y] &= E[e^X] \\ &= \exp(\mu + \frac{1}{2}\sigma^2) \\ VAR[Y] &= E[Y^2] - (E[Y])^2 \\ &= \exp(2(\mu + \sigma^2)) - \exp(2\mu + \sigma^2) \end{aligned}$$

2. We obtain the pdf via the substitution rule.

$$\begin{aligned} X \sim N(\mu, \sigma) & \quad \text{with pdf} f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp(\frac{(x-\mu)^2}{2\sigma^2}) \\ H : y &= e^y \\ H^{-1} : x &= ln(y), \quad DH^{-1} = \frac{1}{y} \\ g(y) &= f(x)\frac{\partial x}{\partial y} \end{aligned}$$

According to the rule of substitution, we get the PDF of Y for all $y \in (0, \infty)$ as

$$\begin{aligned} g(y) &= f(ln(y)) \cdot \frac{1}{y} \\ &= \frac{1}{\sqrt{2\pi\sigma y}} \exp(\frac{(ln(y) - \mu)^2}{2\sigma^2}) \end{aligned}$$

Problem 5 (20 points) In an oral exam a student is asked questions until he or she failed to answer three questions. The student fails the exam once she or he answers more questions incorrectly than correctly. Assume that all questions are equally difficult and independent from each other. The probability for a student to give a correct answer to a question is the same for all questions and equals p. Let Y be the total number of questions asked in such an exam.

1. Give a full definition of the random variable Y.
2. Find the probability mass function of Y.
3. Check that the function you have found in subquestion 2 is indeed a probability mass function.
4. For $p = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ compute the probability that the student fails the exam.

1. $Y : (\Omega, \mathcal{A}, P) \to (S, Pot(S))$ with $S = \{3, 4, 5, \ldots\}$

2. $P(Y=s) = p_Y(s) = \binom{s-1}{2} p^{s-3} \cdot (1-p)^3$
3. Obviously, $p_Y(s) \leq 0$.

$$\begin{aligned}
\sum_{s=3}^{\infty} \binom{s-1}{2} p^{s-3} \cdot (1-p)^3 &= (1-p)^3 \sum_{s=3}^{\infty} \binom{s-1}{2} p^{s-3} \\
&= \sum_{k=0}^{\infty} \binom{k+3-1}{2} p^k (1-p)^3 \quad \text{(substitute } k = s-3\text{)} \\
&= (1-p)^3 \sum_{k=0}^{\infty} \binom{k+2}{2} p^k \quad \text{(neg. binom. series)} \\
&= (1-p)^3 (1-p)^{-3} = 1.
\end{aligned}$$

4. The students fails the exam whenever less than six questions are asked, i.e. $P(\text{exam failed}) = P(Y < 6) = \sum_{s=3}^{5} p_Y(s)$.

Table 1: Probabilities of failing exam for $p = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

p	$1/4$	$1/2$	$3/4$
$P(Y < 6)$	0.8964844	0.5000000	0.1035156

Problem 6 (10 points) Let X have a uniform distribution on $[0, 1]$. Find the probability density function for $Y = X^2$ and prove that the result is a probability density function.

We obtain the pdf via the substitution rule.

$$\begin{aligned}
X \sim U(0,1) & \qquad \text{with pdf} f(x) = \begin{cases} 1 & \textit{for } 0 \leq x \leq 1 \\ 0 & \textit{else.} \end{cases} \\
H : y &= x^2 \\
H^{-1} : x &= \sqrt{y}, \quad DH^{-1} = \frac{1}{2\sqrt{y}} \\
g(y) &= f(x) \frac{\partial x}{\partial y}
\end{aligned}$$

According to the rule of substitution, we get the PDF of Y for all $y \in \mathbb{R}$ as

$$\begin{aligned}
g(y) &= f(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \\
&= \begin{cases} \frac{1}{2\sqrt{y}} & \textit{for } 0 \leq y \leq 1 \\ 0 & \textit{else.} \end{cases}
\end{aligned}$$

Obviously, $g(y)$ is non-negative. Moreover,

$$\begin{aligned}
\int_{-\infty}^{\infty} g(y) dy &= \int_0^1 \frac{1}{2\sqrt{y}} dy \\
&= \sqrt{y}|_0^1 \\
&= 1 - 0 = 0.
\end{aligned}$$

Problem 7 (15 points) A professor has two jars of candy on his desk in which originally N candies were filled in each of them. When a student enters her office the student is invited to choose a jar at random and then take a piece of candy from it. At some time one of the jars will be found empty. At the time when one jar is found empty for the first time, how many pieces of candy are in the other jar? Assume that the student chooses with equal probabilities between the two jars.

Let X be the number of candies in the other jar, once one is found to be empty. The event that $X = x$ means that a total of $2N - x$ draws have taken place before in the $2N - x + 1$th draw one jar is found to be empty. So, in the first $2N - x$ draws we choose with equal probability from the two jars, in the last draw we have to choose the one which is empty. The previous N candies taken from the empty jar can have been taken at any time. Hence we get:

$$P(X = x) = \binom{2N - x}{N} \left(\frac{1}{2}\right)^{2N-x}.$$

To obtain the expected value of candies in the other jar, we use the following recursion:

$$\begin{aligned}
\frac{P(X = x)}{P(X = x - 1)} &= \frac{\binom{2N-x}{N}\left(\frac{1}{2}\right)^{2N-x}}{\binom{2N-(x-1)}{N}\left(\frac{1}{2}\right)^{2N-(x-1)}} \\
&= 2 \cdot \frac{(2N - x)!}{N!(2N - x - N)!} \frac{N!(2N - (x - 1) - N)!}{(2N - (x - 1))!} \\
&= 2 \cdot \frac{N - x + 1}{2N - x + 1}.
\end{aligned}$$

This yields the equation

$$(2N - x + 1)P(X = x) = 2(N - (x - 1))P(X = x - 1).$$

Summing up both sides from $x = 1$ to N yields:

$$\begin{aligned}
& \sum_{x=1}^{N}(2N - x + 1))P(X = x) = 2 \cdot \sum_{x=1}^{N}(N - (x - 1))P(X = x - 1) \\
\Rightarrow\ & \sum_{x=1}^{N}(2N + 1)P(X = x) - \sum_{x=1}^{N} xP(X = x) = \\
& \qquad 2 \cdot \sum_{x=1}^{N} N(P(X = x - 1)) - 2 \cdot \sum_{x=1}^{N}(x - 1)P(X = x - 1) \\
\Rightarrow\ & (2N + 1) \cdot (1 - P(X = 0)) - E[X] = \\
& \qquad 2N \cdot \sum_{x=1}^{N} P(X = x - 1) - 2\left(E[X] - NP(X = N)\right) \\
\Rightarrow\ & E[X] = 2N(1 - P(X = N)) + 2NP(X = N) - 2(N + 1)(1 - P(X = 0)) \\
& \qquad = 2N - (2N + 1) + (2N + 1)P(X = 0) \\
& \qquad = (2N + 1)\binom{2N}{N}\left(\frac{1}{2}\right)^{2N} - 1.
\end{aligned}$$