PSM Fall 2016, Exercise Sheet 5

Return on Friday October 21 in class

Note. You are encouraged to work in teams of two – but no larger. If you work in a team, submit only a single sheet with both names indicated on it. Nicely type-set solutions are highly appreciated. This exercise amounts to 110 points. In case you exceed 100 points, the additional points will be added to other exercises or miniquizzes (if feasible).

Problem 1 (80 points total) Let $(X_1, \ldots, X_n)$ be i.i.d. random variables with known variance σ^2 and unknown mean $\mu \in \mathbb{R}$, which is to be estimated. Assume quadratic loss.

1. (5 points) Formally specify the sample space S, the statistical model $\mathcal{P}$, the decision space D and the loss function L.
2. (20 points) Determine the risk function for a general procedure of the type

$$t(x_1, \ldots, x_n) = a + b\bar{x}_n, \qquad \text{with } a, b \in \mathbb{R}, \bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

   Hint: Note that $E[\big(\mu - (a + b\bar{X}_n)\big)^2] = E[\big((b - (b-1))\mu - a - b\bar{X}_n\big)^2]$

3. (10 points) Derive the specific risk functions for the six procedures $t_{1,n}, t_{2,n}, t_{3,n}, ..., t_{6,n}$ that are defined as follows:

$$\begin{aligned}
t_{1,n} &= \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}_n \\
t_{2,n} &= 2 \cdot \bar{x}_n \\
t_{3,n} &= \frac{\bar{x}_n + \frac{1}{n}}{1 + \frac{1}{n}} \\
t_{4,n} &= -n\bar{x}_n \\
t_{5,n} &= \bar{x}_n + 1 \\
t_{6,n} &= 1.
\end{aligned}$$

4. (10 points) Plot on the same graph the risk functions of these six procedures for $n = 100$ and $\sigma^2 = 1$.
5. (5 points) From these calculations can you assert that any of these six procedures is inadmissible? Which ones?
6. (5 points) If n is large, which of the six procedures would you use, and why?
7. (15 points) Show that $t(x_1, \ldots, x_n) = a + b\bar{x}_n$, with $a, b \in \mathbb{R}, \bar{x}_n = \frac{1}{n}\sum x_i$ is inadmissible for
   (a) $b > 1$
   (b) $b < 0$
   (c) $b = 1$ and $a \neq 0$.
8. (10 points) Suppose the statistician decides to restrict considerations to procedures $t_{a,b,n} = a + b\bar{x}_n$. He or she is concerned about the behaviour of the risk function when μ is large. Show that the risk function remains limited (and actually approaches 0 for large n) as $|\mu| \to \infty$ if and only if $b = 1$; and that *among procedures* with $b = 1$, the choice $a = 0$ gives uniformly smallest risk function.

Problem 2 (30 points)

Instead of a final exam your stats professor offers you a game. He has two coins, one of which is fair, so head and tail occur with probability $\frac{1}{2}$. The other coin is fabricated and comes up heads with probability $\frac{1}{5}$. Your professor selects one coin and you are allowed to make one single toss. You have to come up with a guess whether the coin is fair or fabricated. Assume zero-one loss.

1. (5 points) Provide a formal description of the sample space, the statistical model, its default parameter space, the decision space and the loss function.
2. (5 points) In this context, compute the general form of the risk function for any statistical procedure t.
3. (10 points) Student A now comes up with the idea the professor will always use the fair coin, Student B is convinced that he always uses the fabricated one, Student C and D make their decision dependent on the outcome of the coin toss. So their four statistical procedures are given by

$$
\begin{aligned}
t_A : S \to D, &\qquad x \mapsto d_0, \forall x \in S \\
t_B : S \to D, &\qquad x \mapsto d_1, \forall x \in S \\
t_C : S \to D, &\qquad x \mapsto \begin{cases} d_1, & \text{for } x = 0 \\ d_0, & \text{for } x = 1 \end{cases} \\
t_D : S \to D, &\qquad x \mapsto \begin{cases} d_0, & \text{for } x = 0 \\ d_1, & \text{for } x = 1 \end{cases}
\end{aligned}
$$

   Compute the risk functions for these four procedures.
4. (10 points)
   Compute the Bayes risk of the four statistical procedures for the a priori distribution given by density

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p_\zeta(\theta) = \begin{cases} 0.4 & \mathit{for}\ \theta = \frac{1}{2} \\ 0.6 & \mathit{for}\ \theta = \frac{1}{5}. \end{cases}
$$