PSM Spring 2018, homework 5

1. Let $(S, \mathcal{F})$ be a measurable space and let $A \in \mathcal{F}$ be a nonempty set that is contained in $\mathcal{F}$. Show that the set $\mathcal{F}_A = \{E \cap A \mid E \in \mathcal{F}\}$ is a σ-field on A.

2. Show that the function P' defined in Equation (12) in the LN is indeed a probability measure.

3. (A study of computer hardware reliability). The CPU of a computer contains a large number, say $L = 10,000$, of *logical gates*. Each such gate is an electronic microcircuit which transforms input bits into output bits. The CPU in its entirety is a clocked device – it updates its state at each *clock cycle*. Today's CPUs have clock cycle rates of about 2 GHz, that is, in one second they go through 2,000,000,000 clock cycles. In each clock cycle, all logical gates are executed (this is a simplification, but not crucial for our argument). Digital computers are very sensitive to bit errors that sometimes — very rarely — occur when a logical gate in the CPU misfires in a clock cycle. Computer hardware engineers spend much care on designing gates in a way that leaves only a very small probability for a CPU gate to make an error (it is impossible to get zero error probability due to the laws of quantum mechanics and thermal fluctuations). Compute the error probability that is admissible for a CPU gate if the requirement is that when the computer runs for an entire day, the probability of having zero gate errors is at least 0.999. Hint: use the approximation $log(1 + x) \approx x$ which is good for small x, and which entails $log_{10}(1 + x) \approx x/\log(10) \approx x/2.3$.

Solution for 3.: When the computer is run for 24 hrs, altogether 24 * 3600 * 2,000,000,000 * 10,000 = 1,728,000,000,000,000,000 =: N gate operations are fired. If the probability for a gate error per execution is p, the probability of no gate error in N gate updates is $(1 - p)^N$. We have to solve $0.999 \leq (1 - p)^N$, which leads to $p = 1 - 0.999^{1/N}$ or equivalently, $1 - p = 0.999^{1/N}$. Taking the $\log_{10}$ gives $\log_{10}(1 - p) = (1/N) * \log_{10}(1 - 1/1000)$, hence $-p/2.3 \approx -(1/N) * (1/1000)/2.3$, that is $p \approx 1/(N * 1000) = 1/1,728,000,000,000,000,000,000 \approx 0.58 * 10^{-21}$. In fact, the $\log_{10}$ approximation made on the r.h.s. overestimates by a factor of about 2; but for an order of magnitude estimate this is good enough.

Bit error rates in actual computing devices are far higher (order of 10^{-12} according to a quick web search). These "high" physical bit error rates are compensated by logical hardware designs that automatically detect and correct bit errors; these error corrections mechanisms only fail when certain several bit errors occur simultaneously within a single update cylce which trims the effective error rates into an acceptable range.