Essentials of dynamical systems

H. Jaeger, August 2006

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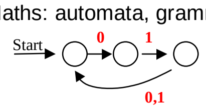
What is a dynamical system?

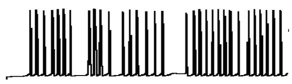
- A DS is any real or artificial system that evolves over time.
- **Symbolic** DS: texts, DNA, robot action sequences, ...
  - Maths: automata, grammars, computer programs, ...



01001...

- **Discrete-time numerical** DS: measurement series, digital signals, ...
  - Maths: iterated maps

$$\mathbf{x}(n+1) = f(\mathbf{x}(n), \mathbf{u}(n)) + v(n)$$



- **Continuous-time numerical** DS: motion of stars, chemical reactions, neural activations, ...
  - Maths: ordinary or partial differential equations (ODE, PDE)

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t)) + v(t)$$

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Modern theory of dynamical systems

Regardless of type of system, some universal topics are of interest:

- Linear vs. nonlinear systems
- Stability vs. Instability
- Stationarity vs. Non-stationarity
- Asymptotic behaviour
- Self-organization, pattern formation
- Oscillations, chaos

The modern mathematical theory of DS has provided many universal concepts and insights.

We concentrate here on continuous-time numerical systems.

Our tutorial example

System equation of a 2-dimensional DS in polar coordinates r and θ:

$$\dot{r} = -r^3 + a\,r$$
$$\dot{\theta} = 1$$

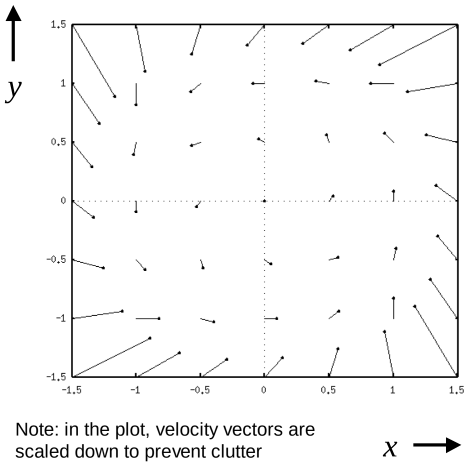
- Read $\dot{r}$ as dr/dt — we describe the time development of r
- Describes a motion of a point (given by coordinates r and θ) in $\mathbb{R}^2$
- r and θ are the state variables. The set of all possible values of the state variables (here: $\mathbb{R}^2$) is the state space (also: phase space)
- a is a (control) parameter, can be set by the "user"; each setting yields a different version of the system equation

Vector fields

$\dot{r} = -r^3 + r$ [we use $a = 1$ for a while]

$\dot{\theta} = 1$

- Use Cartesian coordinates x, y for plotting
- $x = r \cos\theta$, $y = r \sin\theta$
- At each point (x, y) , consider the vector $(\dot{x}, \dot{y})$
- Gives a vector field
- Each vector describes a velocity + direction (velocity field)



Note: in the plot, velocity vectors are scaled down to prevent clutter

For all plots in this section I use Bard Ermentrout's tool XPPAUT5.41 (Google "xppaut", free software)

A closeup look

State ∘ :

$(x, y) = (1,1)$

$(r, \theta) = (\sqrt{2}, \pi/4)$

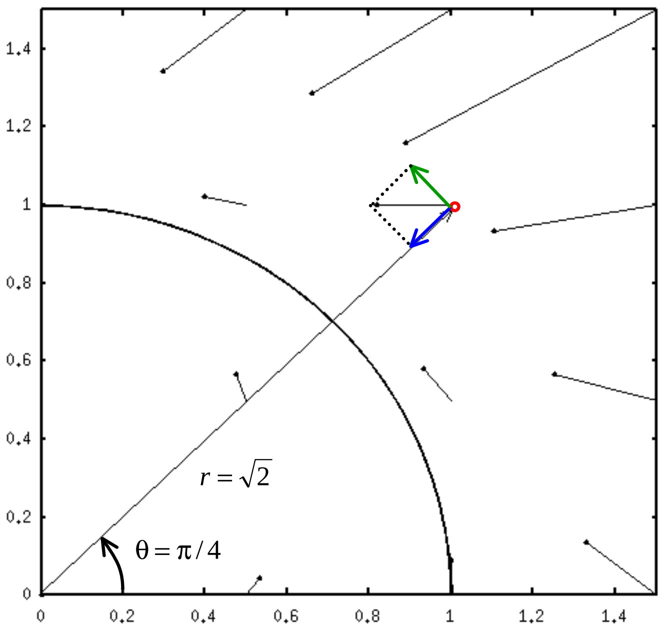
$\dot{r} = -r^3 + r$

$= -\sqrt{2}^3 + \sqrt{2}$

$= -\sqrt{2}$

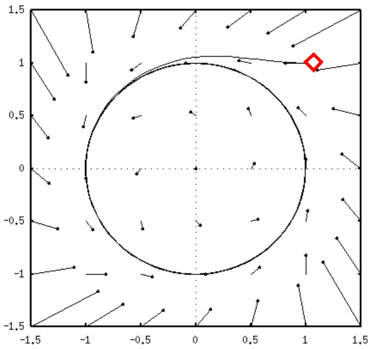
$\dot{\theta} = 1 \Rightarrow$

$\dot{\theta}[(1,1)] = \sqrt{2}$

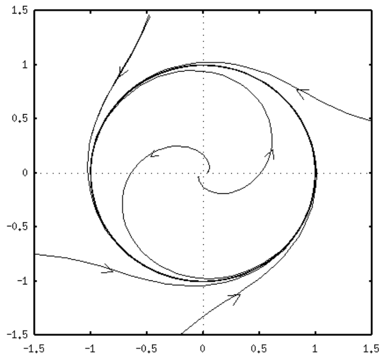


Trajectories and phase portraits

$$\dot{r} = -r^3 + r; \quad \dot{\theta} = 1$$

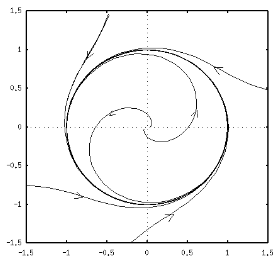


Following the vector field from any starting point ◇ gives a trajectory.



Plotting several characteristic trajectories gives a phase portrait.

Analysis 1



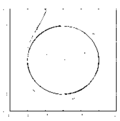
$$\dot{r} = -r^3 + r; \quad \dot{\theta} = 1$$

Observation: (almost) all trajectories home in on a limit cycle. -- Why?

- This example is easy to analyse because the equations for r and θ are decoupled (r does not appear in equation for θ and vice versa).
- $\dot{\theta} = 1$ implies that all motion revolves around the origin with constant angular velocity.
- The key lies in the radial component of the dynamics, $\dot{r} = -r^3 + r$.

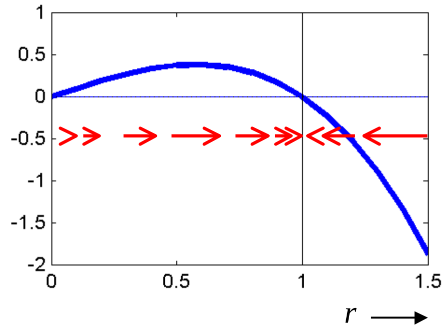
Analysis 2

Analysis of $\dot{r} = -r^3 + r$



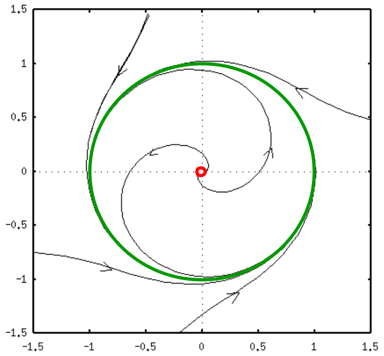
$\dot{r} = -r^3 + r; \quad \dot{\theta} = 1$

$\dot{r} = -r^3 + r$



- For $0 < r < 1$, we have $\dot{r} > 0$, so r will grow with time.
- For $1 < r$, we have $\dot{r} < 0$, so r will shrink with time.
- For $r = 0$ and $r = 1$, we have $\dot{r} = 0$, so r will stay fixed at these values.
- $\rightarrow$ indicate motion of r.

Two special trajectories



$\dot{r} = -r^3 + r; \quad \dot{\theta} = 1$

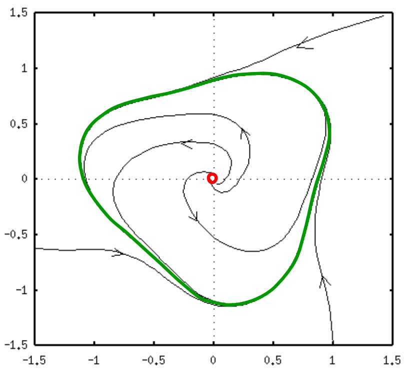
Start from $(0, 0)$: Trajectory will forever remain in $(0, 0)$. This is a fixed point of the dynamics.

This fixed point is unstable: even the smallest perturbation will drive trajectory away from it. The origin is a repellor.

Start from anywhere on the unit circle: the trajectory will forever revolve around the unit circle.

This periodic orbit is stable: when perturbed, the trajectory will asymptotically return to the cycle. It is a limit cycle, a cyclic attractor.

Surprise!

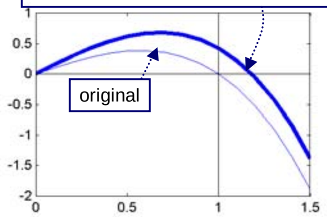


$$\dot{r} = -r^3 + r + 1/2\,\sin(3\theta)\,\sin(r)$$
$$\dot{\theta} = 1 + 1/2\,\sin(4\,r)$$

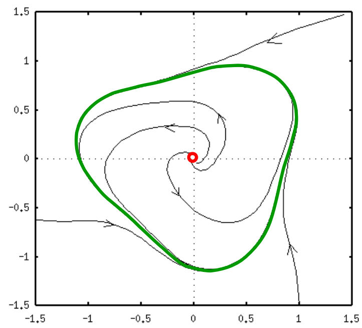
We added some extra terms to the RHS of the system equations.

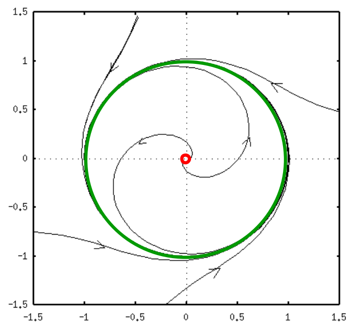
These extra terms "wobble" the original equations, but do not destroy their basic nature.

$$\dot{r} = -r^3 + r + 1/2\,\sin(3\cdot 0.5)\,\sin(r)$$



Structural stability

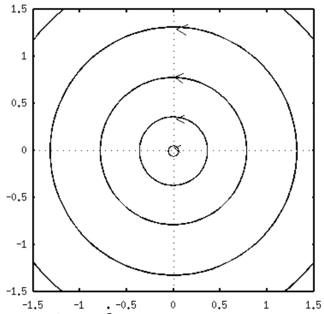




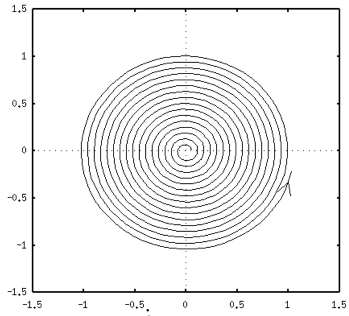
- Two DS are structurally similar if their phase portraits can be continuously transformed into each other.
- Structural similarity is an equivalence relation. It is used to classify DS.

- A DS is structurally stable if small changes to its vector field lead to structurally similar phase portraits.
- Our example system is structurally stable.

Structural instability



$\dot{r} = 0 \quad \dot{\theta} = 1$



$\dot{r} = 0.01 \quad \dot{\theta} = 1$

- This system is structurally not stable.
- Arbitrarily small changes in ist equations will make the circles "miss themselves on their return trip" -- giving spiral portraits instead (inbound or outbound)

- This "spiral" phase portrait is structurally stable.

A question relating to you, maths, and the universe

Given a "random" system equation -- will the system be structurally stable or unstable?

Ich spazierte einsmals im Wald herum meinen eitelen Gedanken Gehör zu geben, da fand ich ein steinern Bildnis liegen in Lebensgröße [...] da fing es an sich zu regen und zu sagen: "Lasse mich mit Frieden, ich bin Baldanders." [...]

[Er] nahm darauf mein Buch, so ich eben bei mir hatte, und nachdem er sich in einen Schreiber verwandelt, schrieb er mir nachfolgende Worte darein: "Ich bin der Anfang und das End, und gelte an allen Orten. Magst dir selbst einbilden wie es einem jeden Ding ergangen, hernach einen Discurs daraus formirn und davon glauben was der Wahrheit ehnlich ist, so hast du was dein narrischer Vorwitz begehret."

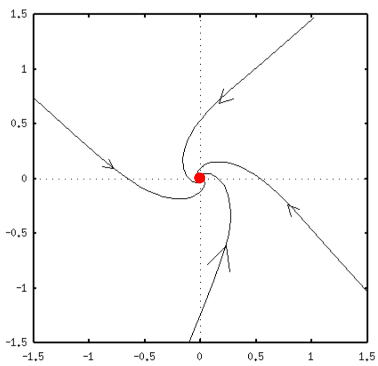
Als er dies geschrieben, wurde er zu einem großen Eichbaum, bald darauf zu einer Sau, geschwind zu einer Bratwurst und unversehens zu einem großen Baurendreck (mit Gunst), er machte sich zu einem schönen Kleewasen, und ehe ich mich versah, zu einem Kuhfladen; item zu einer schönen Blum oder Zweig, zu einem Maulbeerbaum und darauf in einen schönen seidenen Teppich etc. [...] verändert' er sich in einen Vogel, floh schnell davon und ließ mir das Nachsehen.

Grimmelshausen, Simplicius Simplicissimus, sechstes Buch, Kapitel 9

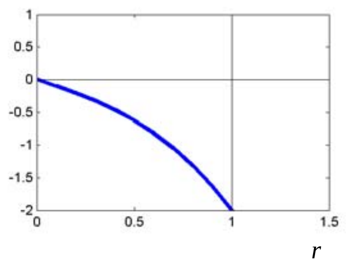
Parameter magic...

$$\dot{r} = -r^3 + a\,r; \quad \dot{\theta} = 1$$

a = –1



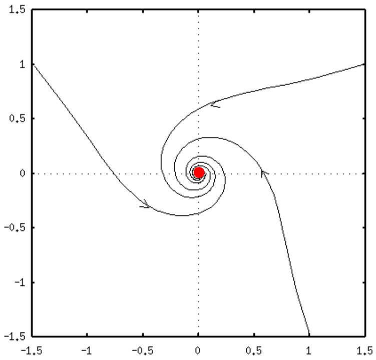
$\dot{r} = -r^3 - 1r$



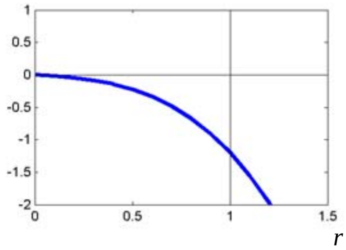
Parameter magic...

$$\dot{r} = -r^3 + a\,r; \quad \dot{\theta} = 1$$

a = –0.2



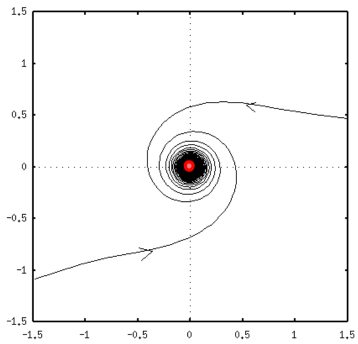
$\dot{r} = -r^3 - 0.2\,r$



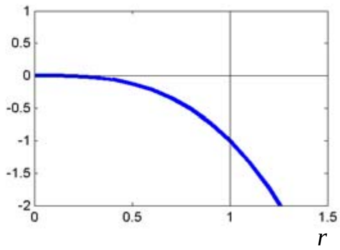
Parameter magic...

$$\dot{r} = -r^3 + a\,r; \quad \dot{\theta} = 1$$

a = 0



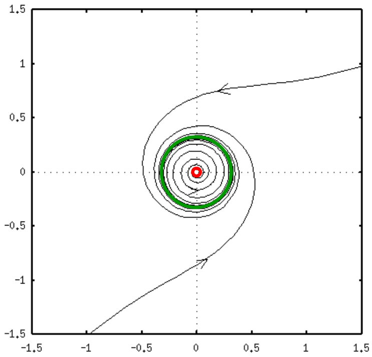
$\dot{r} = -r^3 - 0\,r$



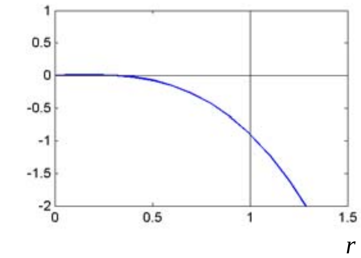
Parameter magic...

$$\dot r = -r^3 + a\,r; \quad \dot\theta = 1$$

a = 0.1



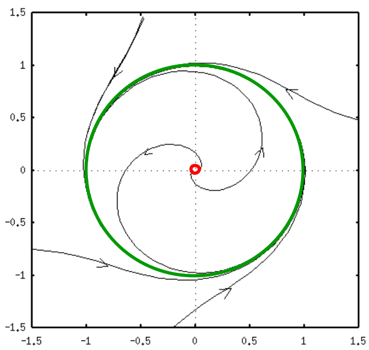
$\dot r = -r^3 + 0.1r$



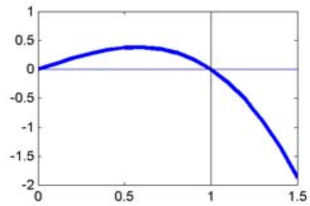
Parameter magic...

$$\dot r = -r^3 + a\,r; \quad \dot\theta = 1$$

a = 1



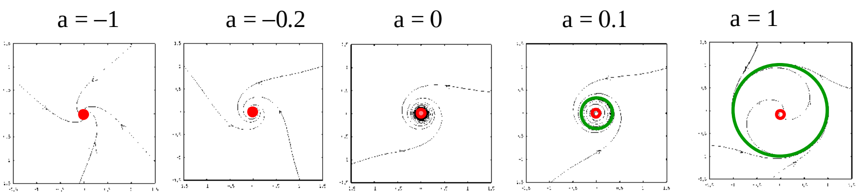
$\dot r = -r^3 + 1r$



r

It's a Bifurcation.

$$\dot{r} = -r^3 + a\,r; \quad \dot{\theta} = 1$$



$a < 0$: Stable fixed point at origin (a point attractor)

Trajectories toward this point have finite length.

System is structurally stable.

$a = 0$: Stable fixed point, infinite trajectories.

Structurally instable.

$a > 0$: Repellor at origin, plus a limit cycle attractor.

Trajectories toward this point have finite length.

System is structurally stable.

Bifurcation: when a control parameter passes through a critical value, the phase portrait changes its nature.

At (and only at) the critical value, the system is structurally instable. "Left" and "right" of this value, the phase portrait is structurally stable.

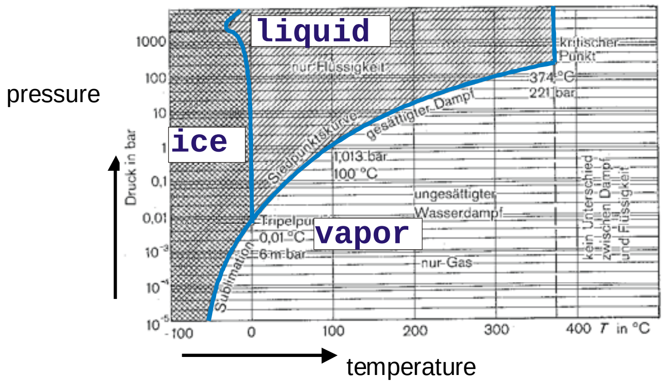
A zoo of bifurcations

Name	"Left" behaviour	Behaviour at critical value	"Right" behaviour
Supercritical Hopf bifurcation	1 point attractor	1 point attractor with "critical slowdown"	1 point repellor, 1 cyclic attractor (whose amplitude grows from zero)
Subcritical Hopf bifurcation	1 point attractor	1 point attractor, 1 cyclic trajectory which is attracting "from left" and repelling "to the right"	1 point attractor, 1 cyclic attractor (which appears out of the blue with nonzero amplitude), 1 cyclic repellor
Pitchfork bifurcation	1 point attractor (or one cyclic attractor)	1 point attractor with "critical slowdown"	2 point attractors (or two cyclic attractors), 1 repellor (or cyclic repellor)
Saddle-node bifurcation	[no attractor or repellor or other fixed point]	1 fixed point, neither attractor nor repellor	1 point attractor, 1 point repellor

... and there are many more ...

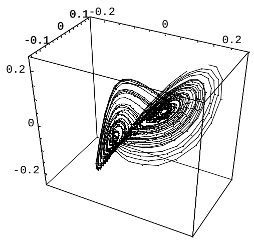
Bifurcations: a universal phenomenon in dynamical systems

Bifurcation: *qualitative* change induced by control parameters

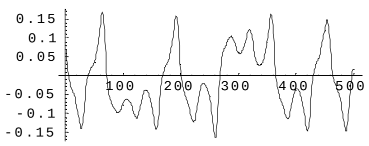


Chaotic attractors

- We have seen point attractors and cyclic attractors.
- Definition (pseudo): An attractor is a subset of points in the state space to which nearby trajectories asymptotically converge.
- There is one more member in the family of attractors: chaotic attractors.
  - The attracting set is fractal.
  - In continuous systems occurs only if dimension is at least 3.
  - Trajectories are non-periodic and unpredictable but deterministic.



The Lorenz attractor (one trajectory shown)



A 1-dim projection of the Lorenz attractor