Solutions to Probability Primer (Midterm Exam)

Algorithmical and Statistical Modelling, Fall 2010.

State whether the following statements are "true" or "false" by giving justification **as briefly as possible** — you are required to either quote a definition or a result (or an elementary fact) or give an example if it can justify/falsify the statement. Points are awarded to a correct certification only depending on the justification that follows it. **Warning :** Pay careful attention to each word in the given statement before you write out your justifications. Also, the justifications could be very simple, one-liners and almost trivial.

1. If $A_1, A_2, \ldots$ are elements belonging to a sigma-algebra $\mathcal{F}$ then the set $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ does not necessarily belong to $\mathcal{F}$.

   False: Let $B_n = \bigcup_{k=n}^{\infty} A_k$. Each $B_n \in \mathcal{F}$ since $\mathcal{F}$ is closed under countable unions. Since $\left(\bigcup_{n=1}^{\infty} B_n^c\right)^c = \bigcap_{i=1}^{\infty} B_n$, and $\mathcal{F}$ is closed under complements, $\bigcap_{i=1}^{\infty} B_n \in \mathcal{F}$.

2. f and g are two nonnegative measurable functions on $(\Omega, \mathcal{F}, \mu)$. If $\int_E f d\mu > \int_E g d\mu$ for every $E \in \mathcal{F}$ then $f(\omega) > g(\omega)$ for every $\omega \in \Omega$.

   The statement is ill framed. There is a typo: "$>$" should have been "$\geq$". All those who have attempted this question have received full points. Now consider the corrected statement: "f and g are two nonnegative measurable functions on $(\Omega, \mathcal{F}, \mu)$. If $\int_E f d\mu \geq \int_E g d\mu$ for every $E \in \mathcal{F}$ then $f(\omega) \geq g(\omega)$ for every $\omega \in \Omega$"

   This statement is false: Suppose $f(\omega) = g(\omega)$ for all $\omega \in A$, where $A \neq \Omega$ and $\mu(A) = \mu(\Omega)$. Also, suppose $f(\omega) < g(\omega)$ for all $\omega \in A^c$. By definition of a Lebesgue integral, $\int_E f d\mu \geq \int_E g d\mu$ for every $E \in \mathcal{F}$.

3. Let X be a real valued random variable. If F is its cumulative distribution function, then F is also the cumulative distribution of any random variable Y, where $Y = X$ almost everywhere.

   True. Let $(\Omega, \mathcal{F}, \mu)$ be the probability space on which X and Y are defined. Then by definition $F(x) = \mu(\{\omega : X(\omega) \in [-\infty, x)\})$. If $Y = X$ almost everywhere, then $\mu(\{\omega : Y(\omega) \in [-\infty, x)\}) = \mu(\{\omega : X(\omega) \in [-\infty, x)\})$ for all $x \in \mathbb{R}$ since $\mu(\{\omega : Y(\omega) \in [-\infty, x)\} \cap \{\omega : X(\omega) \in [-\infty, x)\}) = 0$.

4. A real valued random variable X has a probability density function f_X. Let g be another real valued measurable function defined on $\mathbb{R}$. Then $g(X)$ (i.e., $g(X) = g \circ X$) also has a probability density function.

   False. For the random variable, $g(X)$ to have a probability density function, the probability measure it induces, say $\mathcal{P}_g$ should be absolutely continuous with respect to the Lebesgue measure m, i.e., $m(A) = 0$ implies $\mathcal{P}_g(A) = 0$ for all Borel measurable A. Suppose $g(X)$ takes only values in a set A then $\mathcal{P}_g(A) = 1$. If A is such that $m(A) = 0$, then $\mathcal{P}_g(A)$ is not absolutely continuous with respect to the Lebesgue measure. (Note : for simplicity, in particular, A can be a finite set or a countable collection).

5. The joint cumulative distribution function (cdf) $F_{XY}(x, y)$ of any two uncorrelated random variables X and Y is given by the product $F_{XY}(x, y) = F_X(x)F_Y(y)$, where $F_X(x)$ and $F_Y(y)$ are the cdfs of X and Y respectively.

   False. $F_{XY}(x, y) = F_X(x)F_Y(y)$ is a definition for independence between X and Y. In general, uncorrelatedness is weaker than independence and does not imply independence.

6. Let X and Y be two nonidentical real valued random variables on a probability space with variances greater than zero. Then the variance of $X + Y$ is always greater than zero.

   False. Take $Y = -X$.

7. If (X, Y) is a pair of independent random variables and (Y, Z) is another pair of independent random variables, then X is independent of Z.

   False. Assume $Z = X$ and X is any non-degenerate (a non constant) random variable (a degenerate (constant) random variable is independent with itself, but this finer technicality is not expected).

8. $\frac{1}{n}\sum_{i=1}^{n} X_i \to E[X]$ in probability as $n \to \infty$ for every sequence of random variables $X_1, X_2, \ldots$, where X_i are identically distributed with finite mean $E[X]$ and finite variance σ^2.

   False. Suppose $X_i \equiv X$ for all i and $X_i \not\equiv E[X]$.