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Theory of Input Driven Dynamical Systems

G. Manjunath[1], P. Tiňo[2] and H. Jaeger[3] *

1,3- Jacobs University Bremen, School of Engineering and Science,
28759 Bremen, Germany.
Email:{ m.gandhi, h.jaeger }@jacobs-university.de

2- The University of Birmingham, School of computer Science,
Birmingham B15 2TT, United Kingdom,
Email: P.Tino@cs.bham.ac.uk

Abstract. Most dynamic models of interest in machine learning, robotics, AI or cognitive science are nonautonomous and input-driven. In the last few years number of important innovations have occurred in mathematical research on nonautonomous systems. In understanding the long term behavior of nonautonomous systems, the notion of an attractor is fundamental. With a time varying input, it turns out that for a notion of an attractor to be useful, the attractor cannot a single subset, but must be conceived as a sequence of sets varying with time as well. The aim of this tutorial is to illuminate useful notions of attractors of nonautonomous systems, and also introduce some newly emerging concepts of dynamical systems theory which are particularly relevant for input driven systems.

1 Introduction

Natural systems are subject to time-dependent variations, be it the rhythm of day and night or the yearly seasons or weather patterns that vary from one year to another. At a less macroscopical level, most systems in this world are subject to external forces often varying with time even more irregularly than the weather patterns. Modeling the influence of such time-dependent external forces/influences leads to a mathematical theory of what are called *nonautonomous* dynamical systems[1]. An autonomous system is a dynamical system which has no external input and always evolves according to the same unchanging law, whereas nonautonomous systems are subject to time-varying input or exhibit a temporal change of their update equations. The mathematical theory of nonautonomous systems is considerably more involved than the theory of autonomous systems, and has only recently begun to develop energetically. Since most systems which are of interest in machine learning, robotics, AI, cognitive science are input-driven, it is about time to spread the news about the novel mathematical developments in these fields. In fact, without solid theoretical foundations underpinning input driven dynamical systems we can never hope to obtain deep understanding of the wide variety of dynamical models routinely used in machine learning, robotics, AI, or cognitive science. Moreover, without

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[1]Note that this usage of autonomous/nonautonomous is entirely unrelated to the notion of autonomy as understood in robotics and the agent sciences.

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deep understanding of the workings of the existing models the development of the new ones will not be anchored in a principled framework and will continue to be mostly heuristic based.

Of course, this short tutorial paper cannot aspire to a complete overview of current theoretical developments. Instead, we restrict ourselves to a short introduction to one of the fundamental objects in nonautonomous systems (as fundamental as in the familiar autonomous systems), namely, *attractors*. Attractive sets play prominent role in shaping information latching and transition in input driven dynamical systems. We will also briefly illustrate the difficulty in extending the theory of autonomous dynamical systems to non-autonomous ones by showing how trivial attractor structures in fixed-input regimes can lead to complex state space organizations in the non-autonomous case.

The need for stable dynamical entities like attractors have been desired in both cognitive modeling and artificial (recurrent) neural networks. In cognitive modeling, the notion of an attractor or attractor-like phenomena is based on a scientific metaphor for the brain as a dynamical system capable of hosting large number of stable states which manifest under certain inputs. The studies and usage of artificial recurrent neural networks also point to the existence of stable dynamical phenomenon. References where authors consider such dynamically stable phenomenon in various alternative forms are many. For instance, almost-stable phenomena or a number of alternative "attractor-like" metastability phenomena have been considered that may arise in high-dimensional nonlinear dynamics: saddle point dynamics [17]; attractor relics (or attractor ruins) where classical attractors in a fast-timescale subsystem are destroyed by a slow-timescale saturation dynamics [5]; unstable attractors, a mathematically surprising kind of classical attractors, which however arise generically in certain spiking neural networks and can be left under the impact of arbitrarily small noise because they are surrounded arbitrarily closely by basins of other attractors [20]; high-dimensional attractors (initially named partial attractors) which govern only some dimensions of a high-dimensional phase space [11]; attractor landscapes shaped by control parameter (input) dynamics which lead to the appearance and disappearance of attractors due to incessant bifurcations [14]; coordinated patterns as a basis for pattern generation in motor control [6]; periodic attractors as a basis for a memory and music and rhythm generation [8]; attractors as a basis for a stable working memory [15].

Researchers have used a cautious terminology such as "attractor-like phenomena", when they were aware that traditional dynamical systems theory only provided for attractors in autonomous systems. Often, however, the term "attractor" has been used in an intuitive sense only, without a rigorous meaning.

Much of the literature on nonautonomous dynamical systems focuses on dynamics on unbounded state spaces. Typically, conditions under which a *single*, *global* attractor arises are investigated. This is of limited use in AI, machine learning or robotics, where typically the dynamic models have bounded (even compact) state space and modelers are usually interested in the *multitude* of *local* attractors that arise in such systems. The limited literature on the analysis

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(e.g., [19]) of smaller attractors of bounded systems is inaccessible to a nonexpert and moreover addresses totally invertible systems which again is not a feature of most input driven systems that arise e.g. in neural and cognitive modeling. In this paper we introduce the reader to new notions of attractors which are pertinent for brain-like systems in that they are local phenomena in bounded systems.

We start by providing some background concepts. In this paper, our broad interest will be on discrete-time systems (which automatically arise per discretization in simulations on digital computers). A *discrete-time autonomous system* on a state space X is given by a map $g : X \to X$, where the dynamics is generated by self-compositions (iterations) of g with itself. Any sequence $\{\cdots, x_{-1}, x_0, x_1, \cdots\}$ obtained by $x_n = g(x_{n-1})$ for all $n \geq 1$ is understood to be the evolution of the system. The discrete-time analog of *nonautonomous systems* on a state space X is a family of maps $\{g_n\}$, where each $g_n : X \to X$ is a continuous map, and the state of the system at a time instant n, x_n satisfies $x_n = g_{n-1}(x_{n-1})$.

In this tutorial we are concerned with a particular class of nonautonomous discrete-time dynamical systems called input-driven systems. An *input-driven system* (IDS) on a space X comprises a continuous map $g : U \times X \to X$ and a sequence $\{u_n\}$ which is the input or a driving sequence, where each $u_n \in U$, so that the state x_n of the system at a time instant n satisfies $x_n = g(u_{n-1}, x_{n-1})$. An IDS is a discrete-time nonautonomous system in the above sense if we define $g_n(\cdot) := g(u_n, \cdot)$ [2].

We remark that much of the theory of nonautonomous systems has been influenced strongly by developments in its subarea "random dynamical systems" (see [1] for a comprehensive introduction). A *discrete-time random dynamical system* can be regarded as an IDS where the input u_n is drawn from a stochastic stationary source. The more general theory of nonautonomous systems is comprehensibly described in a recent book by Kloeden and Ramsmussen [10]. Our presentation in this paper differs from the existing literature in the sense that we do not assume the maps g_n to be surjective. In the next section we show that identifying a single subset as an attractor for nonautonomous systems may be inappropriate, and introduce the notion of an attractor for input driven systems.

2 Nonautonomous sets as candidates of attractors

Given any metric space (X, d), and any nonempty subset A, we denote $B_\eta(A) := \{y \in X : d(x, A) < \eta\}$ as the η-neighborhood of A. Here $d(x, A)$ denotes the distance of a point x to a set A, which is the smallest point-to-point distance[3] of x to any point $a \in A$. The n-fold composition of a map $g : X \to X$ with itself is denoted by g^n.

The notion of an attractor in autonomous system is simple and direct, whereas different attractor notions in nonautonomous systems have to be employed to

[2] If the input u_n is a fixed constant for all n, then such an IDS is an autonomous system.
[3] or infimum of point-to-point distances

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satisfactorily explain the asymptotic behavior even in simple systems. We illustrate this with an example. We begin by recalling a well-established definition of an attractor for an autonomous system in discrete time.

Definition 2.1 Let $g : X \to X$ be a continuous map. A subset $A \subseteq X$ is an attractor of a map $g : X \to X$ if A is a nonempty, closed set such that: **(i)** A is invariant, i.e., $g(A) = A$; **(ii)** For any $\epsilon > 0$ there exists a $\delta > 0$ such that, $x \in B_\delta(A) \implies g^n(x) \in B_\epsilon(A)\ \forall\, n \geq 0$; **(iii)** There exists an $\epsilon > 0$ such that if $x \in B_\epsilon(A)$, then $\lim_{n\to\infty} d(g^n(x), A) = 0$.

Let us consider the example where $g(x) = x^2$ on $X = [0, 1]$. It can be verified that the set $\{0\}$ and the whole space $[0, 1]$ are attractors. In verifying that $[0, 1]$ is an attractor, note that that $B_\epsilon([0, 1]) = \{x : d(x, [0, 1]) < \epsilon\} = [0, 1]$. In general, for a surjective map on a compact space, the whole space is always an attractor. However, we observe in the above example, in the case of $[0, 1]$ as an attractor, for every initial condition $x \in (0, 1)$ the evolution $\{x, g(x), g^2(x), \cdots\}$ converges to $\{0\}$ which is an attractor too, and a proper subset of $[0, 1]$. This behavior where the system states in an attractor converge to another attractor subset within it is considered somewhat trivial and uninteresting in dynamical systems theory. We often find that the 'interesting' dynamics takes place in those attractors which do not have any of its proper subsets as attractors. Such attractors are called minimal attractors.

We now turn our attention to nonautonomous systems. We illustrate that defining a single subset as an attractor would be inadequate for nonautonomous systems and then discuss the prospect of defining an attractor for a nonautonomous system to be a sequence of sets.

Consider the simple example of an IDS $g : \{-1, 1\} \times [0, 1] \to [0, 1]$ defined by

$$g(u, x) := \begin{cases} x^2 & : \text{ if } u = -1, \\ 1 - x^2 & : \text{ if } u = 1. \end{cases} \tag{1}$$

For any sequence $\{u_n\}$, which contains both -1 and 1, we have a nonautonomous system. Consider the particular case where $u_n = (-1)^n$ and the nonautonomous system generated by it, i.e.,

$$g_n(x) := \begin{cases} x^2 & : \text{ if } n \text{ is odd}, \\ 1 - x^2 & : \text{ if } n \text{ is even}. \end{cases} \tag{2}$$

The dynamics of this system can be described as follows. Consider the bi-infinite sequence $\{\cdots, 0, 0, 1, 1, 0, 0, 1, 1, \cdots\}$, where the segments $0, 0$ and $1, 1$ alternatively repeat, and $0, 0$ is placed in the $2n^{\text{th}}$ and $2(n + 1)^{\text{th}}$ position. Let A_n denote the n^{th} element of this bi-infinite sequence. It may be verified that $\{A_n\}$ is a state evolution of (2). Furthermore, consider another bi-infinite sequence $\{B_n\}$, $B_n = A_{n+1}$. It may be verified that $\{B_n\}$ too is a state evolution of (2). With some effort, given any state evolution $\{x_n\}$ of (2), one can verify that exactly one of the following holds: **(i)** $d(x_n, A_n) \to 0$ as $|n| \to \infty$. **(ii)** $d(x_n, B_n) \to 0$ as $|n| \to \infty$. In such a scenario it would be inappropriate to call

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a solitary set, say for instance $\{0,1\}$, an attractor, because such an attractor would then take away the time-information describing which states are getting attracted at which instants. However, since any state evolution approaches one of these two sequences $\{A_n\}$ and $\{B_n\}$ it is appropriate to conceive them as two different attractors. Of course, the mathematical definition of an attractor is not yet made, but this simple example illustrates the difficulty or inappropriateness in defining an attractor of (2) as a solitary set. This becomes more apparent if we consider inputs $\{u_n\}$ in (1) that are not periodic, but of a more general nature. In such cases, an attractor $\{A_n\}$ would also be essentially non-periodic.

In general, as the reader may expect, the attractors are not necessarily sequences of singletons, but sequences of sets, where the position of a set in such a sequence determines the time-index. Such a sequence of sets is called a *nonautonomous set*.

3 Attractors for input driven systems

For simple IDS such as (1) it is not difficult to intuitively understand the notion of an attractor. However, mathematically defining it requires more technicalities than defining an attractor of an autonomous system. Moreover, we will observe that since attractors are nonautonomous sets themselves, different attractor notions which give information on the past and future attractive properties of the nonautonomous systems would be needed. We provide these definitions with examples, and also state some results. The proofs can be found in the indicated references.

In writing down prospective cases of attractors of an IDS in (1), we observed that the states of the IDS approached a time-varying attractor that was a sequence of sets as time decreased or increased. In other words, one sequence of sets was approaching another sequence of sets as the time index of the sequences increased or decreased. In order to describe this "approaching" between sequences of sets mathematically, we need the notion of Hausdorff semi-distance.

For nonempty sets $A, B \subset X$ the *Hausdorff semi-distance* from A to B is $dist(A,B) := \sup\{d(x,B) : x \in A\}$. Equivalently, $dist(A,B) := \inf\{\epsilon : A \subset B_\epsilon(B)\}$. Whenever $A \subset B$, $dist(A,B) = 0$ (or more strongly, $A \subset \overline{B} = Closure(B) \Leftrightarrow dist(A,B) = 0$). The term "semi-distance" is due to the fact that, in general, $dist(A,B) \neq dist(B,A)$. The notion of "approaching" of two sequences of sets $\{A_n\}$, $\{B_n\}$ in some space X can be then captured by stating that $lim_{n\to\infty} dist(A_n, B_n) = 0$.

A nonautonomous dynamical system, and hence an IDS can be formulated as what is called a "process". Such formulation helps simplifying the notation when defining attractors.

Definition 3.1 Let $\mathbb{Z}^2_\geq := \{(n,m) : n, m \in \mathbb{Z}, n \geq m\}$. A *process* ϕ on a state space X is a mapping $\phi : \mathbb{Z}^2_\geq \times X \to X$ which satisfies the evolution properties:

(i). $\phi(m,m,x) = x$ for all $m \in \mathbb{Z}$ and $x \in X$,

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(ii). $\phi(n, m, x) = \phi(n, k, \phi(k, m, x))$ for all $m, k, n \in \mathbb{Z}$ with $m \leq k \leq n$ and $x \in X$,

(iii). for given n, m, the map $\phi(n, m, \cdot)$ is continuous on X.

A nonautonomous system $\{g_n\}$ on X generates a process ϕ on X by setting: $\phi(m, m, x) := x$, $\phi(n, m, x) := g_{n-1} \circ \cdots \circ g_m(x)$. Conversely, for every process ϕ on X, there exists a nonautonomous system $\{g_n\}$ defined by: $g_n(x) := \phi(n + 1, n, x)$. Hence, $\phi(n, m, x)$ represents the state of the nonautonomous system at a time instant n, having evolved from time m, when the system was in state x. We will use the terminology of a "process" while referring to a "nonautonomous system" whenever convenient. To define attractors, we need to define invariant sets and orbits or solutions which converge to them. These are recalled in the following definition, where the notation $\phi(n, m, A)$ is understood to be the set $\bigcup_{x \in A} \phi(n, m, x)$.

Definition 3.2 Let ϕ be a process on a space X. An *entire solution* of a process ϕ is a sequence $\vartheta = \{\vartheta_n\}_{n \in \mathbb{Z}}$ such that $\vartheta_m \in X$ for all m and $\phi(n, m, \vartheta_m) = \vartheta_n$ for all $m \leq n$; a nonautonomous set $\mathcal{A} = \{A_n : A_n \subset X\}$ is said to be *ϕ-invariant* if $A_m \subset X$ for all m and $\phi(n, m, A_m) = A_n$ for all $n \geq m$; a nonautonomous set $\mathcal{A} = \{A_n : A_n \subset X\}$ is said to be ϕ *positively invariant* or ϕ *+invariant* if $\phi(n, m, A_m) \subset A_n$ for all $n \geq m$.

Clearly, an entire solution is a ϕ-invariant set, and a ϕ-invariant set is a ϕ +invariant set. A special type of a ϕ-invariant set exists which contains all the essential dynamics of any nonautonomous system. We call this special ϕ-invariant set a "natural association" (see [13] for details).

Definition 3.3 Let ϕ be a process on a compact space X. The sequence $\{X_n\}$ defined by

$$X_n = \bigcap_{m<n} \phi(n, m, X).$$

is called the *natural association* of ϕ on X.

We now provide the idea behind this definition. In the example (2), each map in the nonautonomous system is surjective. However, this is generally not the case. A simple where surjectivity fails is a recurrent neural network with a standard sigmoid mapping. Thus if a function $g_n : X \to X$ is not surjective, it can occur that for any given time-instant, the states of all possible entire solutions at that particular time instant is contained in only a proper subset of X (shown as a gray patch in Figure 1). It is possible to show the existence of a unique sequence $\{X_n\}$, where each X_n is a nonempty closed subset of X such that **(i)** $g_n(X_n) = X_{n+1}$; **(ii)** every entire solution $\{\vartheta_n\}$ is such that $\vartheta_n \in X_n$; **(iii)** for every $x \in X_k$, there is an entire solution $\{\vartheta_n\}$ such that $x = \vartheta_k$. Such a sequence $\{X_n\}$ turns out to be the natural association of ϕ as stated in Proposition 3.1.

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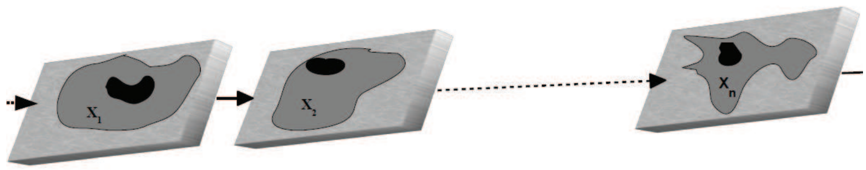

Fig. 1: X is shown as a solid block. All entire solutions pass only through the gray patches, the gray patches being the components X_n of the natural association; the dark patches are components of a local attractor $\{A_n\}$ and $A_n \subset X_n$. A_n attracts only solutions in a neighborhood intersecting X_n but may not attract points outside X_n.

Proposition 3.1 ([13]) *Given a process ϕ on a compact space X, let $\{X_n\}$ be its natural association. Then the following statements hold:* **(i)** *For all $n \geq m$, $\phi(n, m, X_m) = X_n$.* **(ii)** *A nonautonomous set $\mathcal{A} = \{A_k\}$ is ϕ-invariant if and only if for every pair $k \in \mathbb{Z}$, $x \in A_k$ there exists an entire solution $\{\vartheta_n\}$ such that $\vartheta_k = x$ and $\vartheta_k \in A_k$ for all $k \in \mathbb{Z}$-* **(iii)** *A ϕ-invariant set $\{Y_n\}$ is the natural association of ϕ, i.e., $Y_n = X_n$ for all n, if and only every entire solution $\{\vartheta_n\}$ is such that $\vartheta_k \in Y_k$ for all $k \in \mathbb{Z}$.*

In the simple case where all the maps g_n are surjective on X, every component X_n in the natural association is X itself. However, when the the maps are not surjective, the natural association definition is useful and substantial as it contains all the essential dynamics. We incorporate this entity in the definition of attractors.

Given a process ϕ and natural association $\{X_n\}$, we adopt the following notation: for every $A \subset X_i$, we denote

$$B_\eta^{(i)}(A) := B_\eta(A) \cap X_i := \{x \in X_i : d(x, A) < \eta\}.$$

Definition 3.4 Let ϕ be a process on a space X, with the natural association $\{X_n\}$, and let $\mathcal{A} = \{A_n\}$ be a ϕ +invariant set such that each A_n is compact and $\subset X_n$. If for some $\eta > 0$, any of the following conditions

$$\lim_{j\to\infty} \text{dist}(\phi(n, n-j, B_\eta^{(n-j)}(A_{n-j})), A_n) \;=\; 0 \text{ for all } n,$$
$$\lim_{j\to\infty} \text{dist}(\phi(n+j, n, B_\eta^{(n)}(A_n)), A_{n+j}) \;=\; 0 \text{ for all } n,$$
$$\exists N > 0 \lim_{j\to\infty} \text{dist}(\phi(n+j, n, B_\eta^{(n)}(A_n)), A_{n+j}) \;=\; 0 \text{ for all } n > N,$$

holds, then in that order of compliance, $\mathcal{A}$ is respectively called a *local +invariant-pullback-attractor*, *local +invariant-forward-attractor* and *eventual local +invariant-forward-attractor*. If in addition $\mathcal{A}$ is ϕ-invariant then they are also local ϕ invariant attractors or just local attractors of the corresponding types.

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The term "local" is made to signify that a η in the above definition can be arbitrarily small indicating a localized attractive property of any such attractor. For instance, consider the definition of a local +invariant-pullback-attractor made above. The role of the natural association for such an attractor is evident from the $B_\eta^{(*)}$ neighborhood, and the definition merely requires that A_n attracts entire solutions only in a neighborhood intersecting X_n and not (necessarily) anything more (see Figure 1).

At the moment, only the definitions of local ϕ-invariant forward and local pullback attractors, and not the more general case of ϕ + invariant type of attractors, are available in the literature. However, the attractivity principle behind both ϕ-invariant and ϕ +invariant attractors are the same except for the difference in their invariance.

We now briefly explain the definitions of ϕ-invariant local pullback and local forward attractors before indicating the need for the ϕ +invariant attractors.

Pullback attractor. The idea of pullback attraction was introduced in the mid 1990s in the context of random dynamical systems and was subsequently applied to more general nonautonomous systems. To obtain pullback convergence one would have to start progressively earlier at $n-j$ with $j \to \infty$ in order to end up getting arbitrarily closer to A_n. See Figure 2 for a visualization of pullback action. In Figure 2, for instance, the black batch which belongs to $B_\eta^{(n-1000)}(A_{n-1000})$ gets more closely packed near A_n than the gray patch in $B_\eta^{(n-100)}(A_{n-100})$ to A_n at time n.

Forward attractor. In comparison to pullback attractors, forward attractors are intuitively closer to our usual preconceptions about attraction. The principle behind forward convergence is that if one is already sufficiently close to some A_n (regardless of the past) then asymptotically, one gets closer to the components of the forward attractors in the future.

We first consider the nonautonomous system (2). It may be verified that the two sets $\{A_n\}$ and $\{B_n\}$ defined in Section 3 are both pullback and forward attractors. The components of the attractors in this example are all singletons. We next give an example where pullback and forward attractors differ from each other. Consider the IDS $g : \{-1, 1\} \times [0, 1]$ defined by $g(u, x) = \sqrt{x}$ if $u = -1$, $g(u, x) = x^2$ if $u = 1$, with an input sequence $\{u_n\}$ such that $u_n = -1$ for $n < 0$ and $u_n = +1$ for $n \geq 0$. Clearly for any entire solution that does not intersect $\{0, 1\}$, the entire solution approaches $\{0\}$ as $n \to -\infty$ and approaches $\{1\}$ as $n \to \infty$. This aspect is reflected by the local pullback and local forward attractors. It may be verified that $\{A_n\}$, where $A_n = \{0\}$ for all n is a local pullback attractor while $\{A_n\}$, where $A_n = \{1\}$ is a local forward attractor. In the above example, a pullback attractor gives information on the attractivity for the past, while a forward attractor gives information on the attractivity for the future.

In general, there may or may not exist a local forward attractor at all. Consider the nonautonomous system $g(u, x) : [-\frac{3}{2}.\frac{3}{2}] \times [-1, 1] \to [-1, 1]$ given by $g(u, x) = ux/(1 + |x|)$ and u_n being a sequence increasing monotonically such that $\lim_{n\to\infty} u_n = \frac{3}{2}$ and $\lim_{n\to-\infty} u_n = -\frac{3}{2}$. For this system all entire solu-

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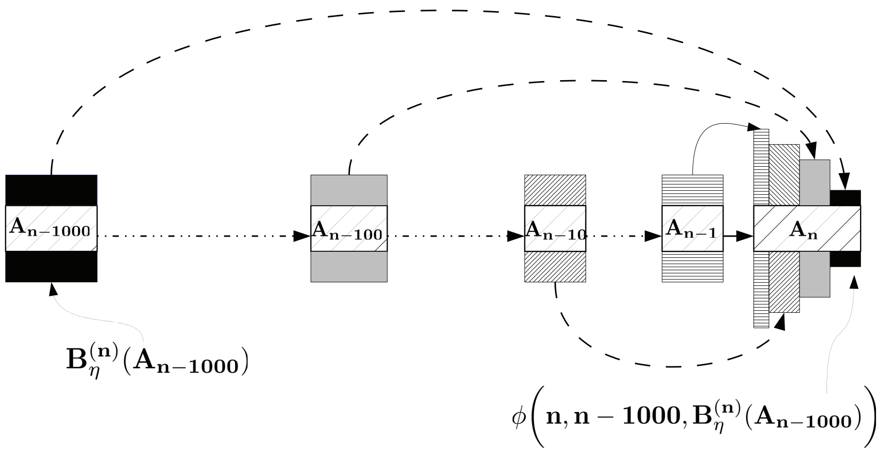

Fig. 2: Pullback action of $\{A_n\}$: A_n's are blocks (with a 45 degree hatching); the different shaded or hatched region around them is the $B_\eta^{(*)}$ neighborhood. The arrowed broken lines from the different shaded or hatched regions are meant to indicate where these regions are mapped to at time n (getting mapped to disjoint sets at time n is for the sake of illustration only). Pullback action is visualized by observing entire solutions passing through the $B_\eta^{(n-j)}$ neighborhood of A_{n-j} get closer to A_n at the time instant n as j increases.

tions which do not intersect $\{0\}$ converge to $\frac{1}{2}$ or to $-\frac{1}{2}$. But there is no entire solution passing through these points $\frac{1}{2}$ or to $-\frac{1}{2}$ and hence they do not belong to any forward attractor. For past attractivity, it may be verified that $\{A_n\}$, where $A_n = \{0\}$ for all n is a pullback attractor .

In general for an IDS $g(u, x)$, with each input u_n belonging to a *finite* set U, there always exists a local forward and local pullback attractor which faithfully reflects the past and future attractivity of every entire solution [13]. The situation when U is infinite is different. There always exists a pullback attractor which reflects the past attractivity of an entire solution but there may not exist any forward attractor which reflects the future attractivity of an entire solution [10]. In such cases the more general ϕ +invariant attractors of the types in Definition 3.4 exist in some cases, and even more general attractors such as the eventual local +invariant-forward-attractors as in (iii) of Definition 3.4 are needed to explain satisfactorily the future behavior of an entire solution [13].

We close the section by noting that it may happen that there exists two distinct attractors $\{A_n\}$ and $\{B_n\}$ of the same type where A_j intersects B_k for some $j \neq k$. This tells us that for a solution to get attracted to a given attractor, it has to be at the right place at the right time in the state space. In the special class of recurrent neural networks called Echo State Networks [7], such an issue does not arise. For such systems, for every given input along with its corresponding IDS, there is an unique attractor which happens to be

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the natural association itself. In general, for lack of additional concepts and machinery we are still unable to identify all the attractors of an IDS.

4 Complexity of state evolution in input driven systems

We conclude the paper by mentioning some interesting observations related to complexity of state evolution in a popular class of nonautonomous dynamical models used in the machine learning community, namely reservoir models, e.g. [7, 12, 21]. Such models are often constructed so that the maps g_n are contractions for all n. In that case, the autonomous dynamics of each individual g_n is extremely simple, governed by a unique attractive fixed point. However, the state evolution when the model is driven by an input stream over U can appear quite “complex”. To quantify the complexity of such state trajectories one may borrow tools from the theory of complex autonomous dynamical systems. In particular, the geometrical complexity of state evolution can be characterized by different kinds of fractal dimensions (e.g. [4]).

On the other hand, if U is finite (and hence can be considered a finite alphabet over which the input streams are formed), a natural notion of complexity of input streams is provided by the topological entropy of the information source emitting the streams. Briefly, topological entropy of an information source is the exponential rate of increase of the number of allowed (e.g. non-zero probability) distinct subsequences of length $\ell \geq 1$ (ℓ-blocks) one can observe in the input streams, as ℓ increases. Obviously, rigid input structures, such periodic streams, have zero topological entropy.

Interestingly enough, one can show that in such models, the complexity of state evolution directly reflects the complexity of the input driving source. In particular, the upper box counting and Haussdorf fractal dimensions of the state evolution form upper and lower bounds, respectively, of the scaled topological entropy of the input source [22]. A more involved multifractal analysis of the state evolution in such input driven systems under a class of stochastic input sources with memory can be found in [22].

These results only underline the main message of this contribution: The usual notions of theory of autonomous dynamical systems are not adequate to capture the richness of the non-autonomous systems. Yet, while such systems are widely used in AI, machine learning or robotics, with new models being continually introduced, a unifying theory of input driven nonautonomous systems has yet to be fully developed. In our previous example, one may ask: Where does the perceived complexity of the state evolution come from? Is it due to the complex nature of the input driving source, or due to the complex autonomous dynamics of the individual maps g_n, or both? Theory of autonomous systems, while profound and deep in many respects, is not suitable for answering such questions.

Another interesting issue is the notion of computation at “the edge of chaos” [2]. When extending numerical calculation of Lyapunov exponents (as signatures of chaos) from autonomous systems to input-driven systems (e.g. [23]), it is

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observed that often enhanced computational power of nonautonomous systems corresponds to regimes with such 'pseudo' Lyapunov exponents close to 0. While the term "edge-of-chaos" can be misleading, since there is no well defined notion of 'chaos' for input driven systems, it nevertheless is an interesting observation and a theory of exactly in what sense and why the edge of stability is important for input driven systems would be a welcome contribution to the research activity in this field.

5 Conclusions

Most dynamic models of interest in machine learning, robotics, AI, cognitive science are nonautonomous and input-driven. The fact that a nonautonomous system depends on the absolute starting time as well as the the time that has elapsed since starting has many deep consequences. Many concepts and results from the autonomous case are no longer valid, or are relevant only for special cases of nonautonomous systems and exclude many interesting types of behavior. One such example is that an attractor in the autonomous case is a single subset, but that would fail to define an attractor in the nonautonomous case. Other developments of nonautonomous dynamical systems such as in linearization theory [16], stability theory [3], bifurcation theory [16] and random dynamics [1] establish a young and active field of research.

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