3.7 Representation of Sequences by Fourier Transforms

Regarding the fact that complex exponential sequences are eigenfunctions of linear time-invariant systems, if we can find the a representation of an arbitrary signal, $x[n]$ as a linear combination of complex exponentials in the form of

$$x[n] = \sum_k a_k e^{j\omega_k n}, \tag{3-44}$$

then from the principle of superposition, we can find the corresponding output of the LTI system if we know the frequency response of the system:

$$y[n] = \sum_k a_k H\left(e^{j\omega_k}\right) e^{j\omega_k n}. \tag{3-45}$$

Now the question is how to represent an arbitrary sequence in the form of Eq. (3-44). A practically useful mathematical tool in this case is the following *Fourier representation* of the input sequence:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega \tag{3-46}$$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{3-47}$$

The synthesis formula in Eq. (3-46) is known as *inverse Fourier transform* which represents $x[n]$ as a superposition of infinitesimally small complex sinusoids of the form:

$$\frac{1}{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega \tag{3-48}$$

where ω ranges over an integral of length 2π and $X\left(e^{j\omega}\right)$ determines the relative amount of each complex sinusoidal component. To determine how much of each frequency component is required to synthesize $x[n]$, Eq. (3-47) provides an expression for computing $X\left(e^{j\omega}\right)$ from $x[n]$. This representation is called *Fourier transform* and it is, in general, a complex valued function of frequency ω.

As with the frequency response, Fourier transform is represented either in rectangular form by its real ($X_R\left(e^{j\omega}\right)$) and imaginary components ($X_I\left(e^{j\omega}\right)$) as:

$$X\left(e^{j\omega}\right) = X_R\left(e^{j\omega}\right) + jX_I\left(e^{j\omega}\right) \tag{3-49}$$

or in polar form via its magnitude ($\left|X\left(e^{j\omega}\right)\right|$) and phase ($\measuredangle X\left(e^{j\omega}\right)$) as:

$$X\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right| e^{j\measuredangle X\left(e^{j\omega}\right)} \tag{3-50}$$

By comparing Eqs. (3-47) and (3-33), it is obvious that the frequency response of a linear time-invariant system is simply the Fourier transform of the impulse response. Therefore, the impulse response can be obtained from the frequency response by applying the inverse Fourier transform integral:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{3-51}$$

It is worth mentioning that the Fourier transform of a discrete signal (e.g., $(e^{j\omega})$) and the frequency response of a LTI system (e.g., $H(e^{j\omega})$) are always periodic functions of the frequency variable ω with period 2π. To show this, we need to prove, for example, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = X(e^{j(\omega+2\pi)})$ in Eq. (3-47). Using the fact that $e^{\mp j2\pi n} = 1$ for an integer, n, we have $e^{-j(\omega+2\pi)n} = e^{-j\omega n}e^{-j2\pi n} = e^{-j\omega n}$. Therefore, $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$ and more generally, $X(e^{j\omega}) = X(e^{j(\omega+2\pi r)})$.

It must be pointed out that Eq. (3-46) is the *inverse* of Eq. (3-47) that is, the Eq. (3-46) compensates exactly the effect of Fourier transform such that if we cascade two systems which respectively calculate the synthesis (Eq. (3-46)) and analysis (Eq. (3-47)) of the sequence $x[n]$, then the output of the system is always equal to the input of the system.

Determining the class of signals that can be represented by Eq. (3-46) is equivalent to considering the convergence of the infinite sum in Eq. (3-47). It can be proved that if x[n] is absolutely summable (i.e., $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$), then $X(e^{j\omega})$ exists. That is, absolute summability is a sufficient condition for the existence of a Fourier transform representation. Since a *stable* sequence is absolutely summable, all stable sequences have Fourier transforms. Consequently, we can show any stable system have a finite and continuous frequency response.

For a class of sequences that can be expressed as sum of discrete frequency components such as,

$$x[n] = \sum_k a_k e^{j\omega_k n}, \quad -\infty < n < \infty \tag{3-52}$$

the Fourier transform representation is defined as:

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty}\sum_k a_k\, 2\pi\delta(\omega - \omega_k + 2\pi r) \ . \tag{3-53}$$

where $\delta(\omega)$ is function of a continuous variable and therefore is of “infinite height, zero width, and unit area,”. Mathematically speaking, for continuous impulse function, i.e,

$$\delta(\omega) = \begin{cases} 0 & \omega \neq 0 \\ inf & \omega = 0 \end{cases}, \tag{3-54}$$

$X(e^{j\omega}) = X(e^{j0})\delta(\omega)$, $\int_{-\infty}^{\infty} \delta(\omega) = 1$ and $\delta(\omega) * X(e^{j\omega}) = X(e^{j\omega})$.

Example 3-5: Fourier transform of a constant

Consider the sequence $x[n] = 1$ for all n. This sequence is neither absolutely summable nor square summable, and Eq. (3-47) does not converge. However, it is possible and useful to define the Fourier transform of the sequence $x[n]$ to be the periodic impulse train described in Eq. (3-53) with $k = 1, a_1 = 1$ and $\omega_1 = 0$:

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi r) \tag{3-55}$$

Example 3-6: Fourier transform of complex exponential sequence, $x[n] = e^{j\omega_0 n}$ for any n

Clearly, $x[n]$ is neither absolutely summable, nor is it square summable. Therefore, we should again represent it as a function of periodic continuous impulse function. Following the Eqs. (3-52) and (3-53), $k = 1, a_1 = 1$ and $\omega_1 = \omega_0$; therefore,

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi r) \tag{3-56}$$

3.8 Symmetry Properties of the Fourier Transforms

Symmetry properties of the Fourier transform are often very useful for simplifying the solution of problems. Before presenting the properties, however, some definitions need to be introduced:

1- If $x[n] = x^*[-n]$, where $*$ denotes the complex conjugation, then the sequence $x[n]$ is called conjugate-symmetric and represented by $x_e[n]$. A real sequence that is conjugate symmetric such that $x[n] = x[-n]$ is called *even* sequence.

2- If $x[n] = -x^*[-n]$, then the sequence $x[n]$ is called conjugate-antisymmetric and represented by $x_o[n]$. A real sequence that is conjugate antisymmetric such that $x[n] = -x[-n]$ is called *odd* sequence.

3- Any sequence $x[n]$ can be expressed as a sum of a conjugate-symmetric and conjugate antisymmetric sequence, i.e.:

$$x[n] = x_e[n] + x_o[n] \tag{3-57a}$$

where,

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) = x_e{}^*[-n] \tag{3-57b}$$

and

$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) = x_o{}^*[-n] \tag{3-57c}$$

A Fourier transform $X(e^{j\omega})$ can be decomposed into a sum of conjugate-symmetric and conjugate-antisymmetric functions as:

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega}) \tag{3-58a}$$

where,

$$X_e(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + X^*(e^{-j\omega})\right) \tag{3-58b}$$

and

$$X_o(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) - X^*(e^{-j\omega})\right) \tag{3-58c}$$

It is obvious that $X_e(e^{j\omega})$ is conjugate-symmetric and $X_o(e^{j\omega})$ is conjugate-antisymmetric.

The symmetry properties of the Fourier transform are summarized in Table 3-1.

Table 3-1 symmetry properties of the Fourier transform of an arbitrary sequence.

	sequence $x[n]$	Fourier transform $X(e^{j\omega})$
1	$x^*[n]$	$X^*(e^{-j\omega})$
2	$x^*[-n]$	$X^*(e^{j\omega})$
3	$Re\ (x[n])$	$X_e(e^{j\omega})$
4	$j\ Im\ (x[n])$	$X_o(e^{j\omega})$
5	$x_e[n]$	$X_R(e^{j\omega}) = Re\ (X(e^{j\omega}))$
6	$x_o[n]$	$X_I(e^{j\omega}) = j\ Im\ (X(e^{j\omega}))$

If $x[n]$ is real and $X(e^{j\omega})$ represents its Fourier transform, then the properties stated in Table 3-2 are applied.

Table 3-2 symmetry properties of the Fourier transform of an arbitrary sequence.

1	$X(e^{j\omega}) = X^*(e^{-j\omega})$
2	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$: the real part is even
3	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$: the imaginary part is odd
4	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$: the magnitude is even
5	$\sphericalangle X(e^{j\omega}) = - \sphericalangle X(e^{-j\omega})$: the phase is odd
6	$X_I(e^{j\omega}) = j\, Im\,(X(e^{j\omega}))$
7	$X_R(e^{j\omega})$ is Fourier transform of the even part of $x[n]$ (i.e., $x_e[n]$)
8	$j\, X_I(e^{j\omega})$ is Fourier transform of the odd part of $x[n]$ (i.e., $x_o[n]$)

Example 3-7: the Fourier transform of the *real* sequence $x[n] = a^n u(n), |a| < 1$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty}(ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega n}} \qquad (3\text{-}59)$$

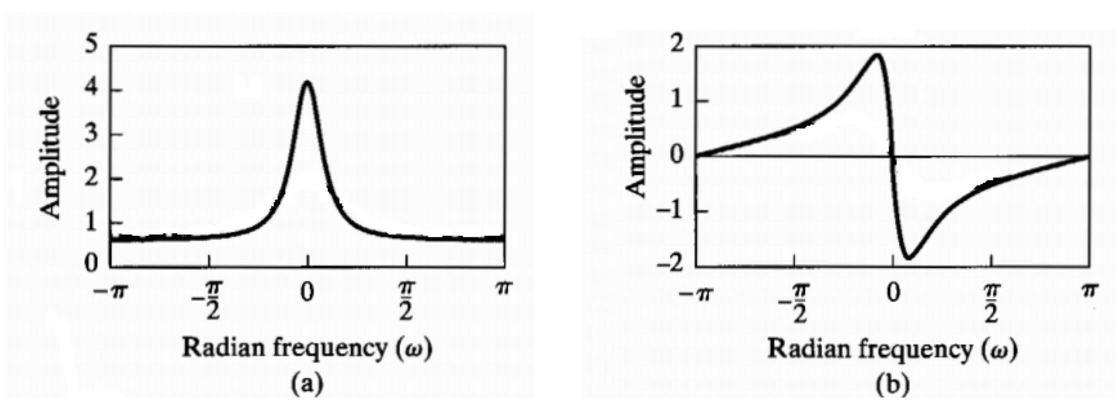
Then from the property 1 in Table 3-2, $X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega n}} = X^*(e^{-j\omega})$. From property 2, $X_R(e^{j\omega}) = \frac{1-a\cos\omega}{1+a^2-2a\cos\omega} = X_R(e^{-j\omega})$. From From property 3, $X_I(e^{j\omega}) = \frac{-a\sin\omega}{1+a^2-2a\cos\omega} = -X_I(e^{-j\omega})$. From properties 4 and 5:

$$\lvert X(e^{j\omega})\rvert = \sqrt{\frac{1}{1+a^2-2a\cos\omega}} = \lvert X(e^{-j\omega})\rvert$$

and

$$\sphericalangle X(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1-a\cos\omega}\right)$$

The real and imaginary parts of $X(e^{j\omega})$ as well as its magnitude and phase are illustrated in Figure 3-9 for $a = 0.9$.



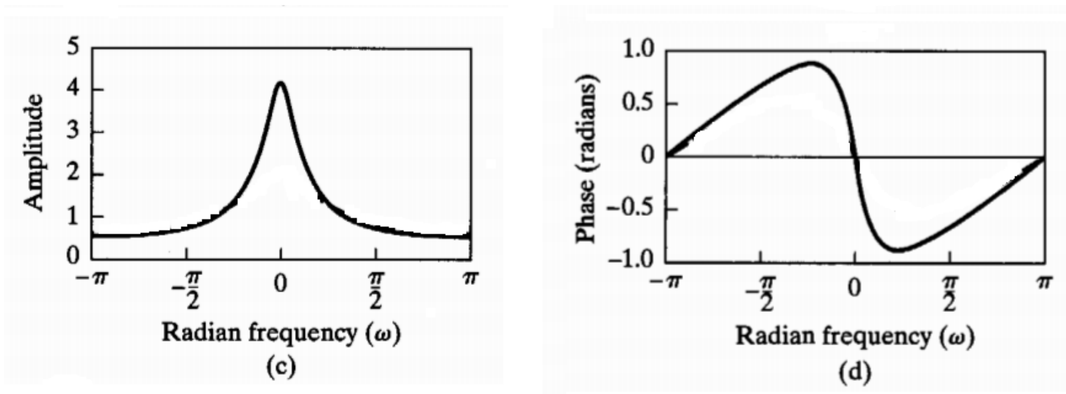


Figure 3-9: Frequecy response for a system with impulse response, $h[n] = a^n u(n), |a| = 0.9$. a) real part, b) imaginary part, c) magnitude and d) phase [1].

3.9 Fourier Transforms Theorems

In addition to the symmetry properties, a different theorems relate operations on the signal to operations on the Fourier transform. These theorems are summarized in Table 3-3 (Although the proof of the theorems are straightforward, we ignore them in this course). To better understand the statements, the following operator notation must be considered:

- $x[n] \overset{\mathcal{F}}{\leftrightarrow} X(e^{j\omega})$ denotes taking the Fourier transform of $x[n]$ such that $X(e^{j\omega}) = \mathcal{F}\{x[n]\}$ and $\mathcal{F}^{-1}$is the inverse operation such that $x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$.

Table 3-3 Fourier transform theorems.

		sequences $x[n], y[n]$	**Fourier transform $X(e^{j\omega}), Y(e^{j\omega})$**
1	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2	Time Shifting	$x[n - n_d], n_d\ integer$	$e^{-j\omega n_d}X(e^{j\omega})$
3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4	Time Reversal	$x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ is real
5	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
6	Convolution Theorem	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7	Windowing Theorem	$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
8	Parseval's theorem	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
9		$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

A number of fundamental Fourier transform pairs are also summarized in Figure 3-10.

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\lvert a \rvert < 1)$	$\frac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\lvert a \rvert < 1)$	$\frac{1}{(1 - ae^{-j\omega})^2}$
7. $\frac{r^n \sin \omega_p (n+1)}{\sin \omega_p} u[n] \quad (\lvert r \rvert < 1)$	$\frac{1}{1 - 2r \cos \omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin \omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c, \\ 0, & \omega_c < \lvert\omega\rvert \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\omega} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

Figure 3-10: a list of useful Fourier transform pairs [1].

Example 3-8: The Fourier transform of $x[n] = a^n u[n-5]$

In order to compute, $X(e^{j\omega})$, we can rewrite the $x[n] = a^5 a^{(n-5)} u[n-5]$ to be able to use the "Time Shifting" property in Table 3-3. Now, let's define $x_1[n] = a^n u[n]$ such that $x[n] = a^5 x_1[n-5]$. From Figure 3-10, the Fourier transform of $x_1[n]$ is $X_1(e^{j\omega}) = \frac{1}{1-e^{-j\omega}}$. Following the time shifting property, $X(e^{j\omega}) = a^5\, e^{-j5\omega} X_1(e^{j\omega}) = a^5\, \frac{e^{-j5\omega}}{1-e^{-j\omega}}$.

Example 3-9: The impulse response of a difference equation $y[n] - \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$

In order to find the impulse response $h[n]$, we need to set $x[n] = \delta[n]$; then, the equation becomes $h[n] - \frac{1}{2}h[n-1] = \delta[n] - \frac{1}{4}\delta[n-1]$. Applying the Fourier transform to both sides of the equation and following the property of "Linearity", we obtain:

$H(e^{j\omega}) - \frac{1}{2}e^{-j\omega}H(e^{j\omega}) = 1 - \frac{1}{4}e^{-j\omega}$ or $H(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$. Now, to obtain $h[n]$, we need to calculate the inverse Fourier transform of $H(e^{j\omega})$. To do this, let's write $H(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} - \frac{\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$. From transformation 4 of Figure 3-10 and the property of time shifting in table 3-3:

$$\mathcal{F}^{-1}\left\{\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right\} = (\frac{1}{2})^n u[n]$$

$$and$$

$$\mathcal{F}^{-1}\left\{\frac{\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}\right\} = -(\frac{1}{4})(\frac{1}{2})^{n-1}u[n-1]$$

Then based on the linearity property:

$$h[n] = (\frac{1}{2})^n u[n] - (\frac{1}{4})(\frac{1}{2})^{n-1}u[n-1]$$

Example 3-10: Inverse Fourier transform of $X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{j\omega})}$

Substitution of $X(e^{j\omega})$ into Eq. (3-46) to determine the inverse transform may lead to an integral that is difficult to evaluate by common integration techniques for real functions. However, applying a simple partial fraction expansion, we can write $X(e^{j\omega})$ as follows:

$$X(e^{j\omega}) = \frac{1}{(1-ae^{-j\omega})(1-be^{j\omega})} = \frac{a/(a-b)}{(1-ae^{-j\omega})} - \frac{b/(a-b)}{(1-be^{j\omega})}$$

Now, from the linearity theorem and using the transform pair 4 in Figure 3-10, we can simply conclude that:

$$x[n] = \frac{a}{(a-b)}a^n u[n] - \frac{b}{(a-b)}b^n u[n]$$