## 5.3 Spectral Analysis

In many clinical and research applications, *power spectral density function* (PSD) of a biomedical signal, known also as "*the spectrum*", is of interest (see Chapter 3 for its definition). Spectral analysis of the EEG can be applied in automatic classification of sleep states, determining the depth of anaesthesia and the classification of a variety of neurological disorders. By processing EMG spectrum the muscle fatigue can be characterized and, to some extent, predicted. Diagnosis of laryngical disorders may benefit from power spectral analysis of speech signals. The analysis of hand tremors, pressure and flow waveforms are another examples of the application of PSD processing in biomedical signals.

The exact PSD function cannot, in general, be calculated, since the given signal is time limited, non-stationary, and corrupted by noise. It is necessary, therefore, to estimate the PSD from the given, short data record. Non-parametric and model-based (parametric) techniques for PSD estimation have been widely investigated and used in EEG processing. In this course, due to the lack of time, only non-parametric, Fourier transform based methods of PSD estimation are introduced. If you are interested, chapter 8 in [1] and chapter 3 in [2] provide a comprehensive insight into parametric/model-based PSD estimation.

Based on its definition, *power spectrum density* of a random signal is Fourier transform of its autocorrelation (See Eq. (3-89) and section 3.12). Therefore, before applying Fourier transformation, the discrete autocorrelation coefficients need to be estimated using the sequence of windowed data. The windowed correlation is then Fourier transformed to provide the estimated PSD. This direct method is known as *Blackman-Tukey*, since it was suggested by Tukey and Blackman. There also exists an indirect approach, known as the *periodogram*, where the PSD estimation is achieved by applying the discrete Fourier transform (DFT) operator directly to the (windowed) data and then smoothing or averaging the absolute values of the DFT. In general, these two methods do not yield identical results, except if a certain biased estimator is used for the correlation estimation, and as many correlation coefficients as data samples are used.

Finite time data sequence is the major bottleneck of the Fourier transform PSD estimation methods. Because the PSD estimate is not that of the process, but that of a sample function, $x[n]$, multiplied by a window. In frequency domain, this yield the Fourier transformation of the signal convolved with Fourier transform of the window and causes side-lobes effects. The most commonly used windows are depicted in Figure 5-13 and are defined by the equations in Table 5-2.

Table 5-2. Commonly used windows in PSD.

| | | |
|---|---|---|
| ***Rectangular*** | $\boldsymbol{w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & otherwise \end{cases}}$ | (5-55a) |
| ***Bartlett (triangular)*** | $w[n] = \begin{cases} 2n/M\,, & 0 \le n \le \frac{M}{2}\,, \\ 2 - 2n/M\,, & \frac{M}{2} \le n \le M, \\ 0, & otherwise \end{cases}$ | (5-55b) |
| ***Hanning*** | $w[n] = \begin{cases} 0.5 - 0.5\cos(2\pi n/M)\,, & 0 \le n \le M, \\ 0, & otherwise \end{cases}$ | (5-55c) |
| ***Hamming*** | $w[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M)\,, & 0 \le n \le M, \\ 0, & otherwise \end{cases}$ | (5-55d) |
| ***Blackman*** | $w[n] = \begin{cases} 0.42 - 0.496\cos(2\pi n/M) + 0.08\cos(4\pi n/M)\,, & 0 \le n \le M, \\ 0, & otherwise \end{cases}$ | (5-55e) |
| ***Kaisar*** | $w[n] = \begin{cases} \dfrac{I_0\left[\beta(1 - [(n-\alpha)/\alpha]^2)^{1/2}\right]}{I_0(\beta)}\,, & 0 \le n \le M, \\ 0, & otherwise \end{cases}$ <br> $\alpha = \frac{M}{2}$ and $I_0(.)$represents the zeroth-order modified Bessel function of the first kind. | (5-55f) |

Fourier transform of these windows are shown in Figure 5-14.

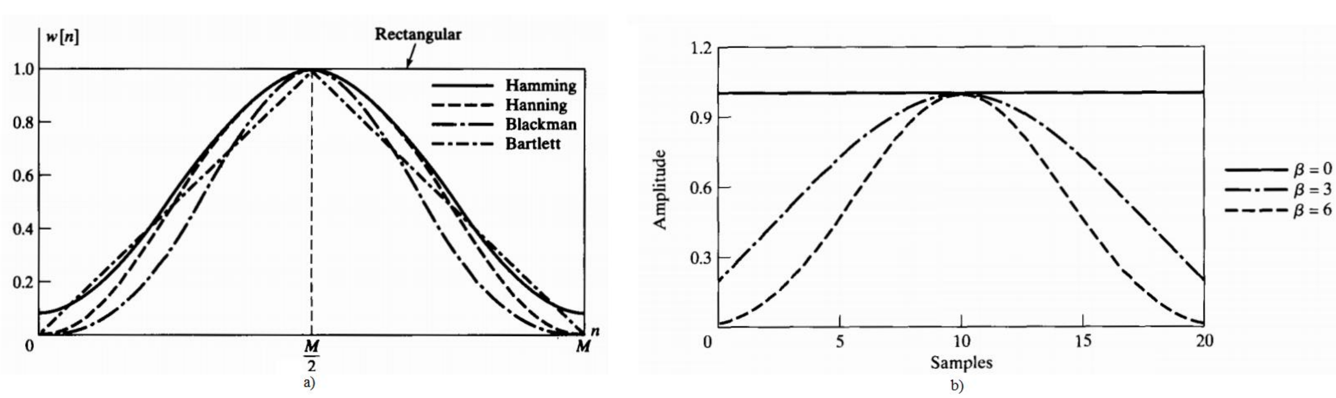


Figure 5-13: a) commonly used DFT windows, b) Kaiser window [Discrete-Time Signal Processing (3rd Edition) (Prentice-Hall Signal Processing Series)].

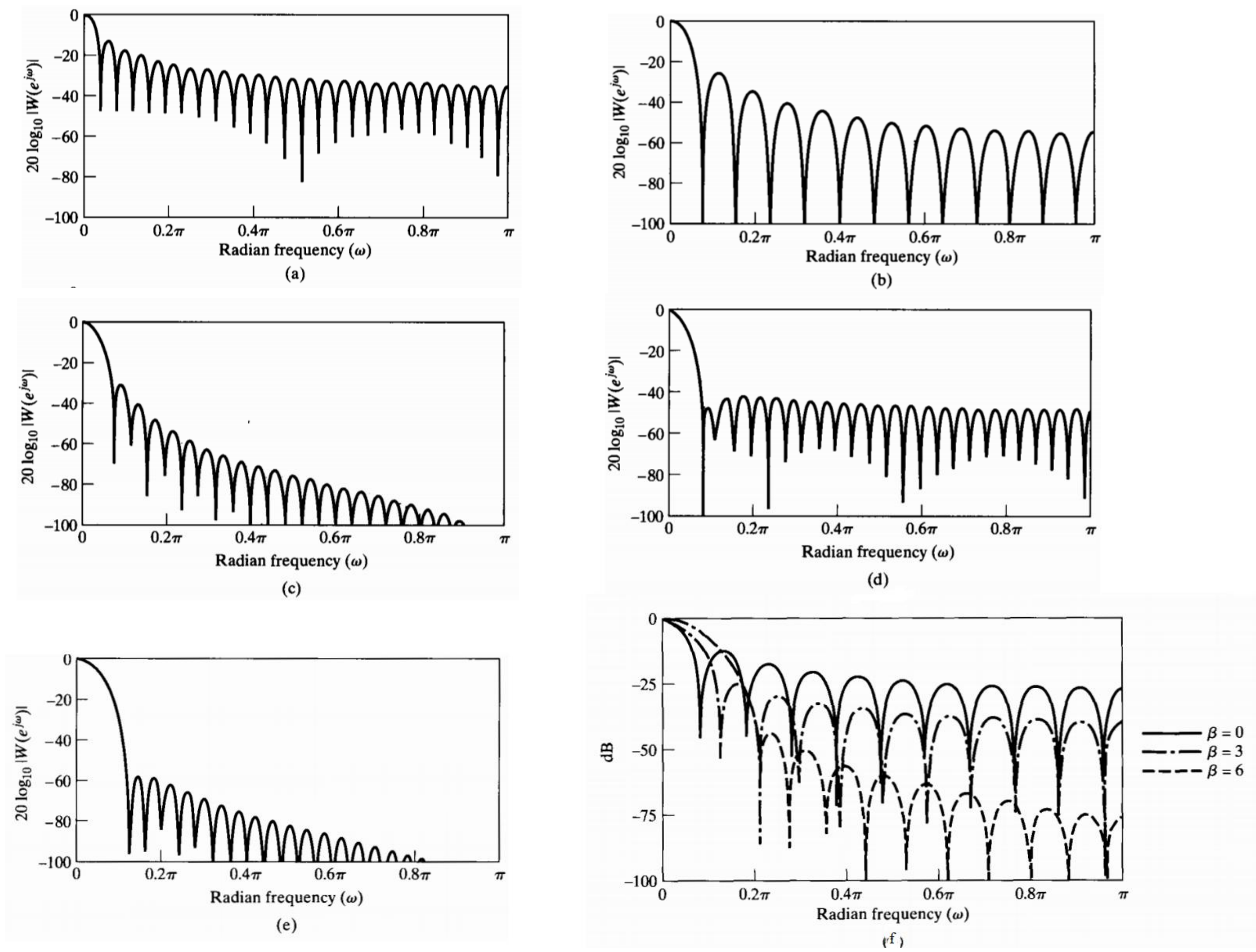


Figure 5-14: Fourier transform (20 log Magnitude) corresponding to windows in Figure 5-13 and Table 5-2 with M = 50 for a) rectangular, b) Bartlett, c) Hanning d) Hamming, e) Blackman and f) Kaiser with M=20 [Discrete-Time Signal Processing (3rd Edition) (Prentice-Hall Signal Processing Series)].

Let's assume the signal has a dominant frequency such that the power of the signal is concentrated in a narrow bandwidth. The convolution operation with one of the frequency responses depicted in Figure 5-14 spreads the power into adjacent frequency bands. This phenomenon is known as *leakage* in signal processing literature. The power of weak sinusoidal components (non-dominant frequencies) in the signal might be completely destroyed by side-lobes of adjacent stronger frequency component. Therefore, the resulting estimation is neither accurate nor discriminative. Two ways to reduce the finite observation time problem are *extrapolation* and *zero padding* techniques. The former estimates the data outside the observation window and latter increases the number of data samples by adding zeroes to the given data sequence. Although the basic resolution of the PSD estimation is not improved by the zero-padding technique, the results are much smoother with less ambiguities in peak spectral determination.

### 5.3.1 The Blackman-Tukey Method

Consider $N$ samples of a data sequence $x[n]$ is available (i.e., $n = 0,1, \dots, N-1$). Its DFT is computed by

$$X[k] = \Delta T \sum_{n=0}^{N-1} x[n] e^{-j(2\pi k)n / N}, k = 0,1, \dots, N-1 \quad (5\text{-}56)$$

where $\Delta T$ is the sampling interval ($\Delta T = \frac{1}{f_s}$ where $f_s$ represents the sampling frequency). The estimation range in the frequency domain is given by:

$$\frac{-\pi}{\Delta T} \leq \omega \leq \frac{-\pi}{\Delta T}. \quad (5\text{-}57)$$

Based on the definition, an estimation of its PSD is:

$$\hat{\Phi}_{xx}[k] = \Delta T \sum_{m=-M}^{M} \hat{\varphi}_{xx}[m] e^{-j\omega m \Delta T}, k = 0,1, \dots, N-1 \quad (5\text{-}58)$$

where $\hat{\varphi}_{xx}[m]$ for $m = -M, \dots, M$ are the discrete estimates of the correlation function. Here are two commonly used correlation estimators:

i. Unbiased estimator

$$\hat{\varphi}_{xx}[m] = \frac{1}{N-m} \sum_{n=0}^{N-m-1} x[n]x[n+m], m = 0,1, \dots, l \leq N-1 \quad (5\text{-}59)$$

ii. Biased estimator

$$\hat{\varphi}_{xx}[m] = \frac{1}{N} \sum_{n=0}^{N-m-1} x[n]x[n+m], m = 0,1, \dots, l \leq N-1$$
$$\hat{\varphi}_{xx}[-m] = \hat{\varphi}_{xx}[m], m = 0,1, \dots, l \leq N-1. \quad (5\text{-}60)$$

The biased estimator of the correlation function has the following expectation:

$$\mathcal{E}\{\hat{\varphi}_{xx}[m]\} = \frac{N-m}{N} \varphi_{xx}[m] \quad (5\text{-}61)$$

which is the true autocorrelation function weighted by a triangular weighting window (Bartlett window). Estimator Eq. (5-61) tends to have lower mean square error than the unbiased one for many finite data sequences. In Blackman- Tukey method for PSD estimation, the biased estimator (i.e., Eq. (5-61)) is used in Eq. (5-58) to estimate the autocorrelation, $\hat{\varphi}_{xx}[m]$. Then, Eq. (5-58) can be solved by means of the FFT. In practice, the sequence, $\hat{\varphi}_{xx}[m]$ used in (5-58) is zero padded to provide a smoother PSD. Figure 5-15 illustrates the Blackman-Tukey estimated PSD of synthesized sine waves corrupted with noise and Figure 5-16 shows the estimation of an EMG signal by the Blackman-Tukey method.

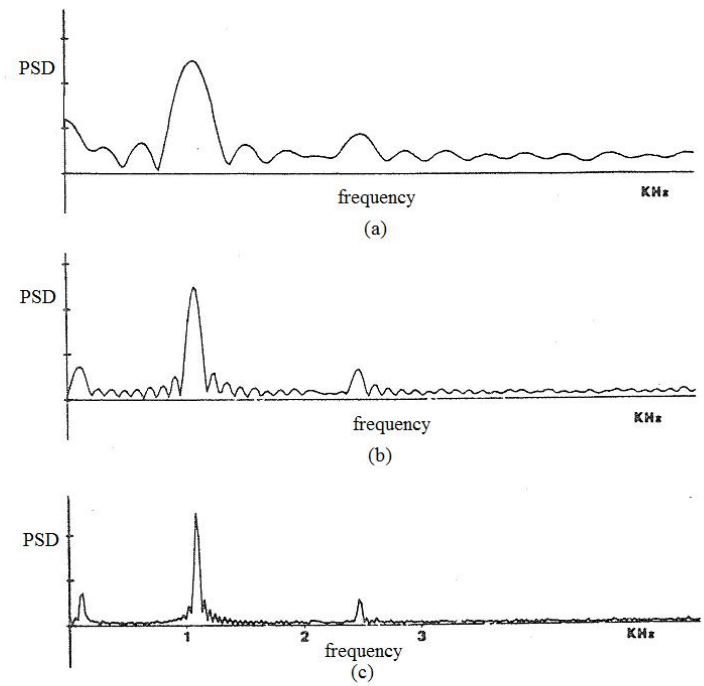


Figure 5-15: Blackman-Tukey estimated PSD of synthesized sine waves corrupted with noise with a) 32 correlation coefficients and 480 padded zeroes, b) 256 correlation coefficients and 416 padded zeroes, and c) 256 correlation coefficients and 256 padded zeroes [1].

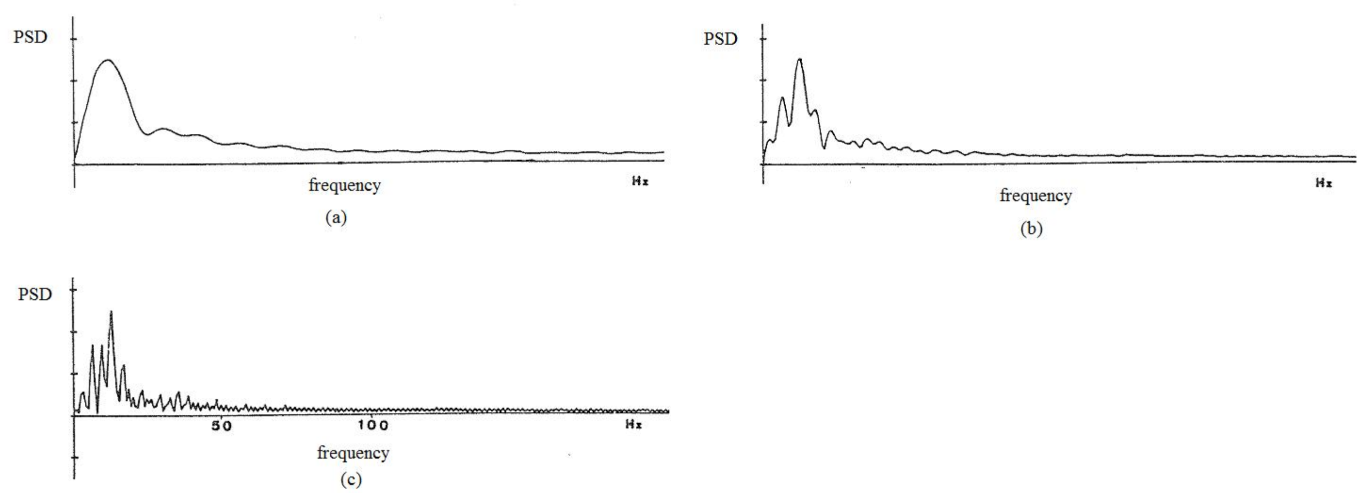


Figure 5-16: Blackman-Tukey estimated PSD of EMG recorded from respiratory diaphragmatic muscle, sampled at 400 Hz with a) 32 correlation coefficients and 480 padded zeroes, b) 256 correlation coefficients and 416 padded zeroes, and c) 256 correlation coefficients and 256 padded zeroes [1].

### 5.3.2 The Periodogram

The periodogram is a method for PSD estimation using the data sequence without the need to first estimate the correlation coefficient. Let's assume that the observed signal (i.e., $x[n]$ for $n = 0,1,\dots,N-1$) is correlation ergodic sampled from a wide-sense stationary continuous signal, $x(t)$; since we have only limited number of samples, we can represent the signal as a windowed data sequence such that

$$x[n] = \begin{cases} w[n]x(n\Delta T), & n = 0,1,\dots,N-1, \\ 0, & otherwise \end{cases} \tag{5-62}$$

where $\Delta T$ is the sampling interval. In this way, we have defined an infinite sequence, hence, we can use infinite correlation estimation, i.e.,:

$$\hat{\varphi}_{xx}[m] = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]x[n+m], -\infty < m < \infty \tag{5-63}$$

Combining the definition of power spectrum density with this time averaging correlation estimate, the following estimate of PSD is obtained:

$$\hat{\Phi}_{xx}[m] = \frac{\Delta T}{N}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x[n]x[n+m]\ e^{-j\omega m\Delta T} \tag{5-64}$$

which can be rewritten as

$$\hat{\Phi}_{xx}[m] = \frac{\Delta T}{N}\sum_{n=-\infty}^{\infty} x[n]\ e^{j\omega n\Delta T}.\sum_{m=-\infty}^{\infty} x[n+m]\ e^{-j\omega(n+m)\Delta T} \tag{5-65}$$

Changing the variable of the inner summation $k = n + m$ we get

$$\hat{\Phi}_{xx}[m] = \frac{1}{N\Delta T}X(e^{j\omega})X^{*}(e^{-j\omega}) = \frac{1}{N\Delta T}\left|X(e^{j\omega})\right|^{2} \tag{5-66}$$

where $X(e^{j\omega})$ is the DFT of $x[n]$ given by Eq. (5-55) and $\left|X(e^{j\omega})\right|^{2}$ is the energy distribution function. The division by $\Delta T$ was required in order to get the PSD. This PSD estimator is known as *periodogram*.

The major advantage of the periodogram method is the fact that one can use the efficient FFT algorithm to compute the DFT. When calculating the square absolute value of the DFT of the sequence $x[n]$ by means of the FFT we get

$$\left|X^{fft}[m]\right|^{2} = \left|\sum_{n=0}^{N-1} x[n]\ e^{-j2\pi mn/N}\right|^{2} \tag{5-67}$$

where $X^{fft}[m]$ is the FFT result which is scaled differently than the quantity calculated from Eq. (5-55). When using the FFT, a scaling factor must be used such that

$$\widehat{\Phi}_{xx}[m] = \frac{\Delta T}{N}\left|X^{fft}[m]\right|^2 \tag{5-68}$$

An estimation of the PSD with periodograms is demonstrated in Figures 5-17 and 5-18. The same signals used for the Blackman-Tukey PSD estimation (Figures 5-15 and 5-16) are used here.

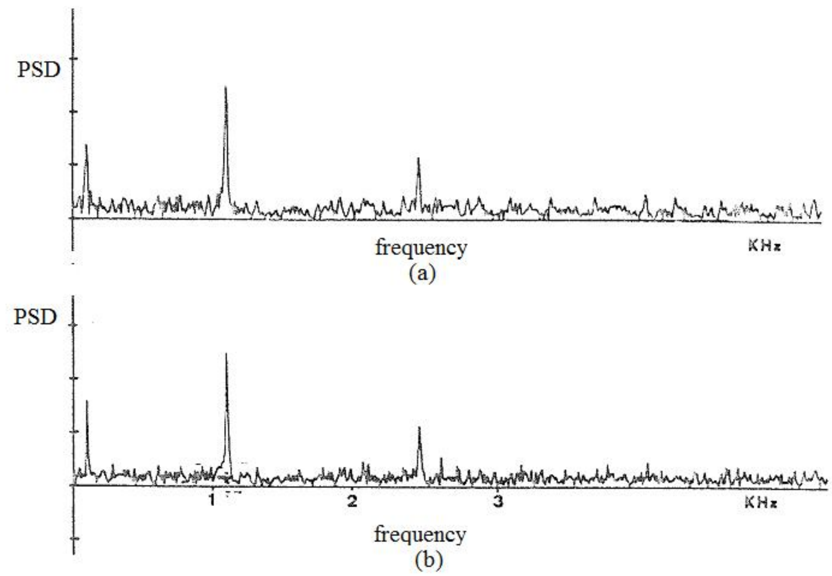


Figure 5-17: Power spectral density function estimation by means of the periodogram. Synthesized noisy sinusoidals as in Figure 5-15 . a) 512 samples and 512 padding zeroes. B) 1024 samples, no padding zeroes [1].

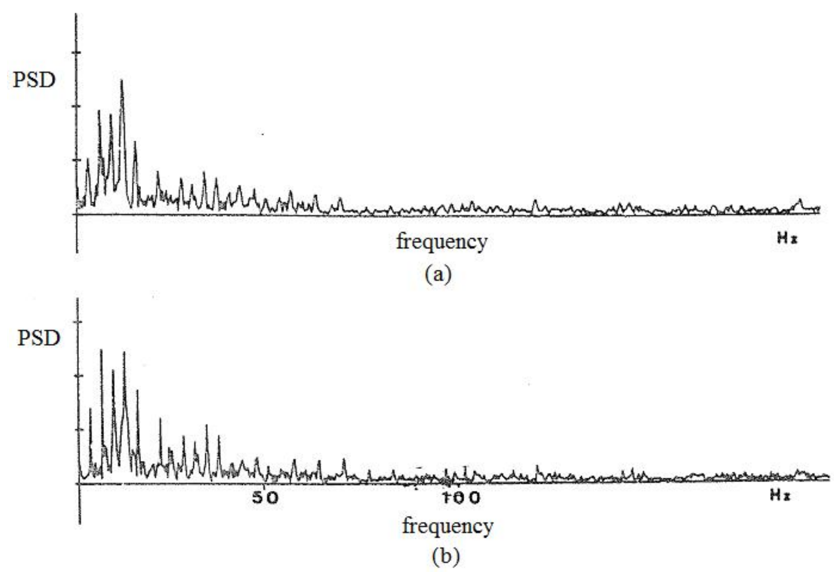


Figure 5-18: Power spectral density function estimation by means of the periodogram. Surface EMG as in Figure 5-16. Traces as in Figure 5-17 [1].

Although the periodogram (Eq. (5-66)) is an efficient estimator from calculations point of view, it is highly biased and has been shown to have large variance. The expected value of this estimator depends on the expected value of correlation estimator (i.e., Eq. (5-63)) and the window in Eq. (5-62). In case of using a rectangular window, as $N$ increases, the window becomes narrower reducing the leakage. At the limit, as $N$ approaches infinity, the window becomes a delta function having no leakage at all.

An effective method for the reduction of the large variance is to average several periodograms calculated from finite segments of the stationary time sequence. An improved method has been suggested by Welch.

Consider the sequence, $x[n]$ is available for $n = 0,1,..,N-1$. We define segments of length $L$, such that the $i$th segment is given by the sequence

$$x^i[n] = x[n + (i-1)D] \quad n = 0,1,\dots,L-1 \quad i = 1,2,\dots,I \tag{5-69}$$

Each two adjacent segments overlap with $D$ samples. The $I$ segments cover the given data sequence $x[n]$ such that the last sample of the $I$'s segment obeys $L + (I - 1)D = N$. We shall now calculate the periodograms of the $I$ overlapped segments, each multiplied by a data window $w[n]$. Denote the normalized periodogram of the $i$th segment by $\widehat{\Phi}^i{}_{xx}[m]$, hence:

$$\widehat{\Phi}^i{}_{xx}[m] = \frac{\Delta T}{E_w L}\left|\sum_{n=0}^{L-1} x^i[n]w[n]\ e^{-j2\pi mn/L}\right|^2 \quad i = 1,2,\dots,I \tag{5-70}$$

Where the normalization factor $E_w$ is the average power of the window:

$$E_w = \frac{1}{L}\sum_{n=0}^{L-1} w^2[n]\ . \tag{5-71}$$

Then the Welch estimate is the average of all $I$ normalized periodograms:

$$\widehat{\Phi}_{xx}[m] = \frac{1}{I}\sum_{i=1}^{I} \widehat{\Phi}^i{}_{xx}[m]\ . \tag{5-72}$$

The expectation of the estimator is similar in nature to the one calculated for the periodogram. The variance, however, is improved.

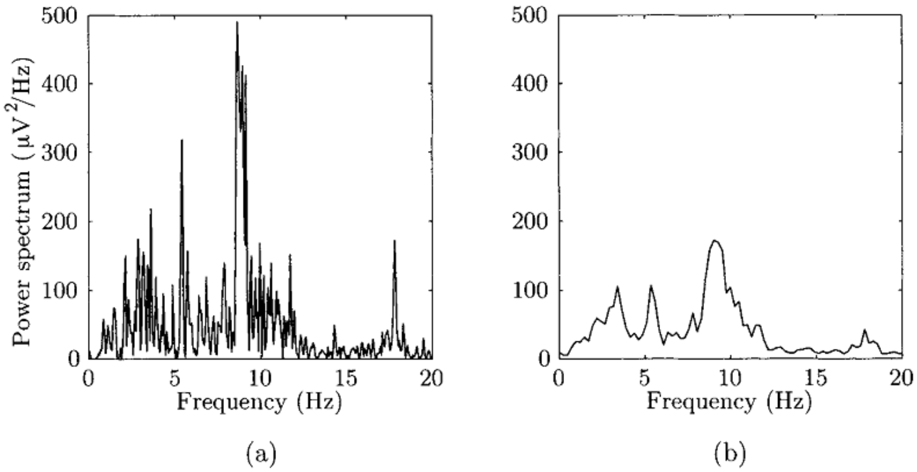


Figure 5-19: Spectral analysis of an EEG with alpha rhythm. (a) The power spectrum obtained without segmentation (N=1024) and (b) with segmentation using I=256 and a segment overlap of 128 samples. The spectral peak related to alpha rhythm is more easily discerned in (b). The EEG signal is the one depicted in Figure 5-3 (a) [2].

An alternative method for reducing the variance of the periodogram is by smoothing. Given a single periodogram, $\widehat{\Phi}_{xx}[m]$ (Eq. (5-66)) calculated from all available data, we can smooth it by passing it through an appropriate spectral filter (window) $H(\omega)$.

Non-parametric spectrum analysis is the substantial component of many EEG analysis systems. It provides a useful interpretation of frequency contents of the signal. However, for quantitative investigations such as classification and statistical analysis, the power spectrum needs to be condensed into a compact set of representative features. This set of parameters describe prominent features of PSD such as peak amplitudes and their respective frequencies, power in relevant frequency bands (alpha, beta, gamma, …), $n$-th order spectral moments and spectral purity index are examples of such representative parameters. Detailed description of this set of parameters is available in chapter 3 of [2].