PSM Spring 2018, homework 3

1. Let $\Omega = \{a, b, c\}$. Define two σ-fields on Ω such that their union is not a σ-field.

2. Let $\Omega = \{a, b, c, d\}$. Let $S = \{1, 2, 3\}$. Construct a nontrivial σ-field $\mathcal{F}$ on S and a σ-field $\mathfrak{A}$ on Ω and a function $X : \Omega \to S$ such that X is $\mathfrak{A}$-$\mathcal{F}$-measurable (a σ-field is trivial if it contains just the empty set and the whole set).

3. Let $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ and $(\Omega_3, \mathcal{F}_3)$ be measurable spaces. If $f_1 : \Omega_1 \to \Omega_2$ and $f_2 : \Omega_2 \to \Omega_3$ are respectively $\mathcal{F}_1$-$\mathcal{F}_2$ and $\mathcal{F}_2$-$\mathcal{F}_3$-measurable functions, prove that $f_2 \circ f_1 : \Omega_1 \to \Omega_3$, where $f_2 \circ f_1(x) := f_2(f_1(x))$ is $\mathcal{F}_1$-$\mathcal{F}_3$-measurable.

4. Let $\varphi : S \to (S', \mathcal{F})$, and let $\mathcal{F} = \sigma(\mathcal{G})$. Show that $\varphi^{-1}(\mathcal{F}) = \sigma(\varphi^{-1}(\mathcal{G}))$.

5. Show that the square function $\mathrm{sqr} : \mathbb{R} \to \mathbb{R},\ \ x \mapsto x^2$ is $\mathfrak{B}(\mathbb{R})$-$\mathfrak{B}(\mathbb{R})$-measurable. Hint: use the previous problem.