PSM Spring 2019, Homework 4 - Solutions

1. Prove the immediate consequences listed in the LN after Definition 7.1.1:

$$\begin{aligned} P(\emptyset) &= 0 \\ P(A^{\mathsf{c}}) &= 1 - P(A) \\ A \subseteq A' &\Rightarrow P(A) \leq P(A') \end{aligned}$$

   Solution.

   $P(\emptyset) = 0$: Use that the intersection $\emptyset \cap M = \emptyset$, that is, technically speaking, $\emptyset$ and M are disjoint. Then by axiom 2, $P(\emptyset) + P(M) = P(\emptyset \cup M) = P(M) = 1$, hence $P(\emptyset) = 0$.

   $P(A^{\mathsf{c}}) = 1 - P(A)$: conclude this from $1 = P(M) = P(A \dot{\cup} A^{\mathsf{c}}) = P(A) + P(A^{\mathsf{c}})$, where the last equality is due to axiom 2.

   $A \subseteq A' \Rightarrow P(A) \leq P(A')$: If $A \subseteq A'$, then $A' = A \dot{\cup} (A' \setminus A)$ and hence $P(A') = P(A) + P(A' \setminus A) \geq P(A)$.

2. Consider the uniform distribution on the unit interval $S = [0, 1]$. Since this is a part of the real line, this sample space is equipped with the Borel σ-field $\mathfrak{B}([0, 1]) = \sigma(\{(a, b) \mid 0 \leq a \leq b \leq 1\})$. For each interval $(a, b]$ in this generator of $\mathfrak{B}([0, 1])$, we have $P(X \in (a, b)) = b - a$. Use this to show that $P(X = a) = 0$ (consider only the case $0 < a < 1$).

   Solution. $(0, 1) = (0, a) \dot{\cup} \{a\} \dot{\cup} (a, 1)$, hence $1 = P((0, 1)) = P((0, a)) + P(\{a\}) + P((a, 1)) = a + P(\{a\}) + 1 - a = P(\{a\}) + 1$, from which $P(X = a) = 0$ follows.

3. Show that

$$P(X \in A, Y \in B, Z \in C) = P(X \in A)\; P(Y \in B \mid X \in A)\; P(Z \in C \mid X \in A, Y \in B).$$

   Solution. Eqn. 7.4 from the LN is equivalent to

$$P(X \in A, Y \in B) = P(X \in A \mid Y \in B)\, P(Y \in B).$$

   By twofold application of this equation, conclude

$$\begin{aligned} P(X \in A, Y \in B, Z \in C) &= P(Z \in C \mid X \in A, Y \in B)\; P(X \in A, Y \in B) \\ &= P(Z \in C \mid X \in A, Y \in B)\; P(X \in A)\; P(Y \in B \mid X \in A) \end{aligned}$$